

Viscosity Solutions of Fully Nonlinear Parabolic Path Dependent PDEs: Part I

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Abstract

The main objective of this paper and the accompanying one [10] is to provide a notion of viscosity solutions for fully nonlinear parabolic path-dependent PDEs. Our definition extends our previous work [8], focused on the semilinear case, and is crucially based on the nonlinear optimal stopping problem analyzed in [9]. We prove that our notion of viscosity solutions is consistent with the corresponding notion of classical solutions, and satisfies a stability property and a partial comparison result. The latter is a key step for the wellposedness results established in [10]. We also show that the value processes of path-dependent stochastic control problems are viscosity solutions of the corresponding path dependent dynamic programming equation.

Key words: Path dependent PDEs, Second order Backward SDEs, nonlinear expectation, viscosity solutions, comparison principle.

AMS 2000 subject classifications: 35D40, 35K10, 60H10, 60H30.

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1 Introduction

The objective of this paper is to introduce a notion of viscosity solution of the following fully nonlinear path-dependent partial differential equation:

$$-\partial_t u(t, \omega) - G(t, \omega, u(t, \omega), \partial_\omega u(t, \omega), \partial_{\omega\omega}^2 u(t, \omega)) = 0, \quad 0 \leq t < T, \quad \omega \in \Omega, \quad (1.1)$$

where the unknown u is a progressively measurable process on the canonical space $\Omega = \{\omega \in C([0, T], \mathbb{R}^d) : \omega_0 = 0\}$, and the nonlinearity $G : [0, T] \times \Omega \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}^d \rightarrow \mathbb{R}$ is progressively measurable, satisfies convenient Lipschitz and continuity assumptions, and is degenerate elliptic.

The latter question attracted our attention after the point raised by Peng in [21] that this would be an alternative approach to the theory of backward stochastic differential equations, introduced by the seminal paper of Pardoux and Peng [17].

The semilinear case, corresponding to the case where G is linear in the $\partial_{\omega\omega}^2 u$ -variable, was addressed in [8], where existence and uniqueness results are established for a new notion of viscosity solution. The main difficulty is related to the fact that the canonical space fails to be locally compact, so that many tools from the standard theory of viscosity solutions do not apply to the present context. The main contribution of [8] is to replace the pointwise extremality in the standard definition of viscosity solutions by the corresponding extremality in the context of an optimal stopping problem under a nonlinear expectation \mathcal{E} . More precisely, we introduce a set of smooth test processes φ which are tangent from above or from below to the processes of interest u in the sense of the following nonlinear optimal stopping problems

$$\sup_{\tau} \overline{\mathcal{E}}[(\varphi - u)_{\tau}], \quad \inf_{\tau} \underline{\mathcal{E}}[(\varphi - u)_{\tau}], \quad \text{where} \quad \overline{\mathcal{E}} := \sup_{\mathbb{P} \in \mathcal{P}} \mathbb{E}^{\mathbb{P}}, \quad \underline{\mathcal{E}} := \inf_{\mathbb{P} \in \mathcal{P}} \mathbb{E}^{\mathbb{P}},$$

τ ranges over a convenient set of stopping times, and \mathcal{P} is a weakly compact collection of probability measures, motivated by a convenient linearization of the nonlinearity F . Consequently, in the particular semilinear case of [8], the family \mathcal{P} consists of equivalent probability measures.

In this paper, we extend the definition introduced in [8] to the fully nonlinear case. In this context, the family \mathcal{P} of probability measures consists of nondominated mutually singular measures. The analysis of the optimal stopping problem under the corresponding nonlinear expectation problem is a major difficulty which is addressed in [9]. Given this, we prove that our definition of viscosity solution is consistent with the corresponding notion of classical solutions, and we show that the value function of path-dependent stochastic

control problems as well as second order backward stochastic differential equations [3, 26] are naturally viscosity solutions of the corresponding path-dependent partial differential equation. This extends the context of backward stochastic differential equations of [8].

We also prove two important properties. First our newly introduced notion of viscosity solution satisfies a stability property similar to the finite-dimensional context. Second, the partial comparison result holds true. Namely, for any pair of viscosity subsolution \underline{u} and supersolution \bar{u} with $\underline{u}_T \leq \bar{u}_T$ on Ω , we have $\underline{u} \leq \bar{u}$ whenever either one of them is smooth. This result is crucial for the well-posedness results established in our accompanying paper [10]. We remark that Peng [22] also investigated the comparison principle for fully nonlinear PPDEs by using a different approach.

We emphasize that our results hold for the general degenerate elliptic case. In particular, our context covers first order path-dependent PDEs, where our notion of viscosity solutions in fact reduces to that of Lukoyanov [13].

The rest of the paper is organized as follows. In Section 2, we introduce the general framework, we define a notion of classical differentiability which is weaker than that of [7]. In Section 3, we introduce our notion of viscosity solution of fully nonlinear PPDE, and we provide various examples which highlight the analogy with the properties of viscosity solutions in finite dimensional spaces. We prove the consistency with the notion of classical solution. In Section 4, we show that natural examples as the value function of path dependent stochastic control problems, or solutions of second order backward stochastic differential equations, are viscosity solutions of the corresponding path-dependent PDEs. Section 5 contains our stability and partial comparison results. Section 6 shows that our framework includes backward stochastic PDE by a convenient augmentation of the canonical space. Finally, Section 7 revisits the semilinear case and provides an alternative and simpler well-posedness argument to that of our previous paper [8] which will be extended to the fully nonlinear case in our accompanying paper [10].

2 Preliminaries

2.1 The canonical spaces

Let $\Omega := \{\omega \in C([0, T], \mathbb{R}^d) : \omega_0 = \mathbf{0}\}$, the set of continuous paths starting from the origin, B the canonical process, \mathbb{F} the filtration generated by B , \mathbb{P}_0 the Wiener measure, and $\Lambda := [0, T] \times \Omega$. Here and in the sequel, for notational simplicity, we use $\mathbf{0}$ to denote vectors or matrices with appropriate dimensions whose components are all equal to 0. Let

\mathbb{S}^d denote the set of $d \times d$ matrices, and

$$x \cdot x' := \sum_{i=1}^d x_i x'_i \text{ for any } x, x' \in \mathbb{R}^d, \quad \gamma : \gamma' := \text{Trace}[\gamma \gamma'] \text{ for any } \gamma, \gamma' \in \mathbb{S}^d.$$

We define a norm on Ω and a metric on Λ as follows: for any $(t, \omega), (t', \omega') \in \Lambda$,

$$\|\omega\|_t := \sup_{0 \leq s \leq t} |\omega_s|, \quad \mathbf{d}_\infty((t, \omega), (t', \omega')) := |t - t'| + \|\omega_{\cdot \wedge t} - \omega'_{\cdot \wedge t'}\|_T. \quad (2.1)$$

Then $(\Omega, \|\cdot\|_T)$ and $(\Lambda, \mathbf{d}_\infty)$ are complete metric spaces. We shall denote by $C^0(\Lambda)$ (resp. $UC(\Lambda)$) the collection of all progressively measurable processes which are continuous (resp. uniformly continuous) in (t, ω) under \mathbf{d}_∞ . The corresponding subsets of bounded processes are denoted $C_b^0(\Lambda)$ and $UC_b(\Lambda)$. Finally, $C^0(\Lambda, \mathbb{R}^d)$ denote the space of \mathbb{R}^d -valued processes with entries in $C^0(\Lambda)$, and we define similar notations for the spaces C_b^0 , UC , and UC_b .

Definition 2.1 By $\underline{\mathcal{U}}$, we denote the collection of all \mathbb{F} -progressively measurable processes $u : \Lambda \rightarrow \mathbb{R}$ such that u is bounded from above and

- for all $\omega \in \Omega$, $t \mapsto u(t, \omega)$ is càdlàg with positive jumps,
- there exists a modulus of continuity function ρ such that for any $(t, \omega), (t', \omega') \in \Lambda$:

$$u(t, \omega) - u(t', \omega') \leq \rho\left(\mathbf{d}_\infty((t, \omega), (t', \omega'))\right) \text{ whenever } t \leq t'. \quad (2.2)$$

By $\overline{\mathcal{U}}$ we denote the set of all processes u such that $-u \in \underline{\mathcal{U}}$.

Remark 2.2 (i) The progressive measurability of u implies that $u(t, \omega) = u(t, \omega_{\cdot \wedge t})$, and it is clear that $\underline{\mathcal{U}} \cap \overline{\mathcal{U}} = UC_b(\Lambda)$. We also recall from [9] Remark 3.2 the redundancy in the above assumption that Condition (2.2) implies that X has left-limits and $X_{t-} \leq X_t$ for all $t \in (0, T]$.

(ii) For simplicity in this paper we consider bounded and uniformly continuous solutions only. We leave for future work the question of replacing the boundedness assumption by some growth condition, and the uniform regularity by some local uniform regularity. ■

We next introduce the shifted spaces. Let $0 \leq s \leq t \leq T$.

- Let $\Omega^t := \{\omega \in C([t, T], \mathbb{R}^d) : \omega_t = \mathbf{0}\}$ be the shifted canonical space; B^t the shifted canonical process on Ω^t ; \mathbb{F}^t the shifted filtration generated by B^t , \mathbb{P}_0^t the Wiener measure on Ω^t , and $\Lambda^t := [t, T] \times \Omega^t$.

- Define $\|\cdot\|_s$ on Ω^t , \mathbf{d}_∞ on Λ^t , and $C^0(\Lambda^t)$ etc. in the spirit of (2.1) and Definition 2.1.
- For $\omega \in \Omega^s$ and $\omega' \in \Omega^t$, define the concatenation path $\omega \otimes_t \omega' \in \Omega^s$ by:

$$(\omega \otimes_t \omega')(r) := \omega_r \mathbf{1}_{[s, t)}(r) + (\omega_t + \hat{\omega}'_r) \mathbf{1}_{[t, T]}(r), \quad \text{for all } r \in [s, T].$$

- Let $s \in [0, T]$ and $\omega \in \Omega^s$. For an \mathcal{F}_T^s -measurable random variable ξ , an \mathbb{F}^s -progressively measurable process X on Ω^s , and $t \in (s, T]$, define the shifted \mathcal{F}_T^t -measurable random variable $\xi^{t,\omega}$ and \mathbb{F}^t -progressively measurable process $X^{t,\omega}$ on Ω^t by:

$$\xi^{t,\omega}(\omega') := \xi(\omega \otimes_t \omega'), \quad X^{t,\omega}(\omega') := X(\omega \otimes_t \omega'), \quad \text{for all } \omega' \in \Omega^t.$$

It is clear that, for any $(t, \omega) \in \Lambda$ and any $u \in C^0(\Lambda)$, we have $u^{t,\omega} \in C^0(\Lambda^t)$. The other spaces introduced before enjoy the same property.

Finally, we denote by \mathcal{T} the set of \mathbb{F} -stopping times, and $\mathcal{H} \subset \mathcal{T}$ the subset of those hitting times H of the form

$$H := \inf\{t : B_t \in O^c\} \wedge t_0 = \inf\{t : d(\omega_t, O^c) = 0\} \wedge t_0, \quad (2.3)$$

for some $0 < t_0 \leq T$, and some open and convex set $O \subset \mathbb{R}^d$ containing $\mathbf{0}$. The set \mathcal{H} will be important for our optimal stopping problem, which is crucial for the comparison and the stability results, see Remark 2.5. We note that $H = t_0$ when $O = \mathbb{R}^d$. Moreover,

$$H > 0, H \text{ is lower semicontinuous, and } H_1 \wedge H_2 \in \mathcal{H} \text{ for any } H_1, H_2 \in \mathcal{H}.$$

Define \mathcal{T}^t and \mathcal{H}^t in the same spirit. For any $\tau \in \mathcal{T}$ (resp. $H \in \mathcal{H}$) and any $(t, \omega) \in \Lambda$ such that $t < \tau(\omega)$ (resp. $t < H(\omega)$), it is clear that $\tau^{t,\omega} \in \mathcal{T}^t$ (resp. $H^{t,\omega} \in \mathcal{H}^t$).

2.2 Capacity and nonlinear expectation

For all $L > 0$ and $t \in [0, T]$, let \mathcal{P}_L^t denote the set of probability measures \mathbb{P} on Ω^t such that there exist \mathbb{F}^t -progressively measurable \mathbb{R}^d -valued processes $\alpha^\mathbb{P}$, an \mathbb{S}^d -valued process $\beta^\mathbb{P}$, and a d -dimensional \mathbb{P} -Brownian motion $W^\mathbb{P}$ satisfying:

$$|\alpha^\mathbb{P}| \leq L, \quad 0 \leq \beta^\mathbb{P} \leq \sqrt{2L}I_d, \quad dB_t = \beta_t^\mathbb{P} dW_t^\mathbb{P} + \alpha_t^\mathbb{P} dt, \quad \mathbb{P}\text{-a.s.} \quad (2.4)$$

As in Denis, Hu and Peng [6], the set \mathcal{P}_L^t induces the following capacity:

$$\mathcal{C}_t^L[A] := \sup_{\mathbb{P} \in \mathcal{P}_L^t} \mathbb{P}[A], \quad \text{for all } A \in \mathcal{F}_T^t. \quad (2.5)$$

We denote by $\mathbb{L}^1(\mathcal{F}_T^t, \mathcal{P}_L^t)$ the set of all \mathcal{F}_T^t -measurable r.v. ξ with $\sup_{\mathbb{P} \in \mathcal{P}_L^t} \mathbb{E}^\mathbb{P}[|\xi|] < \infty$. The following nonlinear expectation will play a crucial role:

$$\overline{\mathcal{E}}_t^L[\xi] := \sup_{\mathbb{P} \in \mathcal{P}_L^t} \mathbb{E}^\mathbb{P}[\xi] \quad \text{and} \quad \underline{\mathcal{E}}_t^L[\xi] := \inf_{\mathbb{P} \in \mathcal{P}_L^t} \mathbb{E}^\mathbb{P}[\xi] = -\overline{\mathcal{E}}_t^L[-\xi] \quad \text{for all } \xi \in \mathbb{L}^1(\mathcal{F}_T^t, \mathcal{P}_L^t). \quad (2.6)$$

We remark that $\overline{\mathcal{E}}_t^L[\xi]$ can be viewed as the solution of a Second Order BSDEs (2BSDE, for short), or a conditional G -expectation. See Section 4 for more details.

Definition 2.3 Let X be a progressively measurable process with $X_t \in \mathbb{L}^1(\mathcal{F}_t, \mathcal{P}_L)$. We say that X is a $\bar{\mathcal{E}}^L$ -supermartingale (resp. submartingale, martingale) if, for any $(t, \omega) \in \Lambda$ and any $\tau \in \mathcal{T}^t$, $\bar{\mathcal{E}}_t^L[X_\tau^{t, \omega}] \leq$ (resp. $\geq, =$) $X_t(\omega)$.

We now state an important result for our subsequent analysis. Given a bounded progressively measurable process X , consider the nonlinear optimal stopping problem

$$\bar{\mathcal{S}}_t^L[X](\omega) := \sup_{\tau \in \mathcal{T}^t} \bar{\mathcal{E}}_t^L[X_\tau^{t, \omega}] \quad \text{and} \quad \underline{\mathcal{S}}_t^L[X](\omega) := \inf_{\tau \in \mathcal{T}^t} \underline{\mathcal{E}}_t^L[X_\tau^{t, \omega}], \quad (t, \omega) \in \Lambda. \quad (2.7)$$

By definition, we have $\bar{\mathcal{S}}^L[X] \geq X$ and $\bar{\mathcal{S}}_T^L[X] = X_T$. The following nonlinear Snell envelope characterization is proved in [9].

Theorem 2.4 Let $X \in \underline{\mathcal{U}}$ be bounded, $\mathbb{H} \in \mathcal{H}$, and set $\hat{X}_t := X_t \mathbf{1}_{\{t < \mathbb{H}\}} + X_{\mathbb{H}-} \mathbf{1}_{\{t \geq \mathbb{H}\}}$. Define

$$Y := \bar{\mathcal{S}}^L[\hat{X}] \quad \text{and} \quad \tau^* := \inf\{t \geq 0 : Y_t = \hat{X}_t\}.$$

Then $Y_{\tau^*} = \hat{X}_{\tau^*}$, Y is an $\bar{\mathcal{E}}^L$ -supermartingale on $[0, \mathbb{H}]$, and an $\bar{\mathcal{E}}^L$ -martingale on $[0, \tau^*]$. Consequently, τ^* is an optimal stopping time.

Remark 2.5 We emphasize that the maturity of the above nonlinear optimal stopping problem is restricted to be a hitting time in \mathcal{H} . This requirement is due to technical aspects in the proof of Theorem 2.4 reported in [9]. The difficulty is related to the approximation of the optimal stopping policy τ^* by continuous random variables in view of the application of the monotone convergence theorem under nonlinear expectation.

2.3 The derivatives

For $s \in [0, T]$, $u \in C^0(\Lambda^s)$, we define its right time-derivative, if it exists, as in Dupire [7]:

$$\partial_t u(t, \omega) := \lim_{h \downarrow 0} \frac{u(t+h, \omega_{\cdot \wedge t}) - u(t, \omega)}{h}, \quad t \in [s, T) \quad \text{and} \quad \partial_t u(T, \omega) := \lim_{t \uparrow T} \partial_t u(t, \omega). \quad (2.8)$$

We define the space derivatives via the functional Itô formula, which plays an important role in our paper. Denote

$$\mathcal{P}_\infty := \bigcup_{L > 0} \mathcal{P}_L \quad \text{and} \quad \mathcal{P}_\infty^t := \bigcup_{L > 0} \mathcal{P}_L^t, \quad t \in (0, T].$$

Definition 2.6 We say $u \in C^{1,2}(\Lambda)$ if $u \in C^0(\Lambda)$, $\partial_t u \in C^0(\Lambda)$, and there exist $\partial_\omega u \in C^0(\Lambda, \mathbb{R}^d)$, $\partial_{\omega\omega}^2 u \in C^0(\Lambda, \mathbb{S}^d)$ such that, for any $(s, \omega) \in [0, T) \times \Omega$ and any $\mathbb{P} \in \mathcal{P}_\infty^s$, $u^{s, \omega}$ is a local \mathbb{P} -semimartingale and it holds:

$$du^{s, \omega} = (\partial_t u)^{s, \omega} dt + (\partial_\omega u)^{s, \omega} \cdot dB^s + \frac{1}{2} (\partial_{\omega\omega}^2 u)^{s, \omega} : d\langle B^s \rangle, \quad \mathbb{P}\text{-a.s.} \quad (2.9)$$

We define $C^{1,2}(\Lambda^t)$ similarly. It is clear that, for any (t, ω) and $u \in C^{1,2}(\Lambda)$, we have $u^{t, \omega} \in C^{1,2}(\Lambda^t)$.

By a direct localization argument, we see that the above $\partial_\omega u$ and $\partial_{\omega\omega}^2 u$, if they exist, are unique. Consequently, we call them the first order and second order space derivatives of u , respectively.

Remark 2.7 (i) The typical case that $\partial_\omega u, \partial_{\omega\omega}^2 u$ exist is the case that u is smooth in Dupire's sense [7], and in that case our space derivatives agree with the vertical derivatives introduced therein, due to the functional Itô formula. Therefore, any smooth function in the sense of Dupire's calculus is also smooth in the sense of Definition 2.6. So our space $C^{1,2}(\Lambda)$ is a priori larger than the corresponding space in [7]. In particular, Definition 2.6 is different from the corresponding definition in our previous paper [8].

(ii) Unlike Dupire's definition, our definition does not require all the derivatives to be bounded, and does not involve any extension of u to a larger domain $[0, T] \times \mathbb{D}([0, T])$, where \mathbb{D} is the set of càdlàg paths. Moreover, we do not need (2.9) to hold true for all semimartingale measures \mathbb{P} .

(iii) In general, our space derivatives are not defined pathwise and we do not require that $\partial_\omega(\partial_\omega u) = \partial_{\omega\omega}^2 u$, although this holds true in typical cases. ■

Example 2.8 Let $d = 1$. As highlighted by Cont and Fournie [4], a simple example of non-differentiable process is the running maximum process: $u(t, \omega) := \bar{\omega}_t := \max_{0 \leq s \leq t} \omega_s$, $(t, \omega) \in \Lambda$. It is obvious that $\partial_t u(t, \omega) = 0$ for all $(t, \omega) \in \Lambda$. However, u is not differentiable in the sense of Definition 2.6. Indeed, if it is differentiable, then one must have $\partial_\omega u = 0$, and $\frac{1}{2} \partial_{\omega\omega}^2 u dt = d\bar{B}_t$, which is impossible under \mathbb{P}_0 . In terms of the Dupire's vertical derivatives, we have $\partial_\omega u(t, \omega) = 0$ whenever $\omega_t < \bar{\omega}_t$, and

$$\partial_\omega^+ u(t, \omega) = 1 \quad \text{and} \quad \partial_\omega^- u(t, \omega) = 0 \quad \text{whenever} \quad \omega_t = \bar{\omega}_t,$$

where ∂_ω^+ and ∂_ω^- denote the right and left space derivatives in the sense of Dupire. Hence the process u is not differentiable on $\{\omega_t = \bar{\omega}_t\}$. ■

We next introduce a localized version of Definition 2.6. For $\mathbf{h} \in \mathcal{H}$. Denote

$$\Lambda(\mathbf{h}) := \{(t, \omega) \in \Lambda : t < \mathbf{h}(\omega)\} \quad \text{and} \quad \bar{\Lambda}(\mathbf{h}) := \{(t, \omega) \in \Lambda : t \leq \mathbf{h}(\omega)\}. \quad (2.10)$$

Then clearly $\Lambda(\mathbf{h})$ is an open subset of $(\Lambda, \mathbf{d}_\infty)$ and, for $u : \bar{\Lambda}(\mathbf{h}) \rightarrow \mathbb{R}$, $(t, \omega) \in \Lambda(\mathbf{h})$, the time derivative $\partial_t u(t, \omega)$ is well defined by (2.8), whenever the limit exists.

Definition 2.9 Let $\mathbf{H} \in \mathcal{H}$ and $u : \bar{\Lambda}(\mathbf{H}) \rightarrow \mathbb{R}$. We say

- (i) $u \in C^0(\Lambda(\mathbf{H}))$ if u is continuous in $(t, \omega) \in \Lambda(\mathbf{H})$ under \mathbf{d}_∞ ,
- (ii) $u \in C^0(\bar{\Lambda}(\mathbf{H}))$ if $u \in C^0(\Lambda(\mathbf{H}))$, and $u(\cdot, \omega)$ is continuous on $[0, \mathbf{H}(\omega)]$ for all $\omega \in \Omega$,
- (iii) $u \in C^{1,2}(\bar{\Lambda}(\mathbf{H}))$ if $u \in C^0(\bar{\Lambda}(\mathbf{H}))$, $\partial_t u \in C^0(\Lambda(\mathbf{H}))$, and there exist $\partial_\omega u \in C^0(\Lambda(\mathbf{H}), \mathbb{R}^d)$, $\partial_{\omega\omega}^2 u \in C^0(\Lambda(\mathbf{H}), \mathbb{S}^d)$ such that (2.9) holds in $\Lambda(\mathbf{H})$ \mathbb{P} -a.s. for all $(s, \omega) \in \Lambda(\mathbf{H})$ and $\mathbb{P} \in \mathcal{P}_\infty^s$.

By a direct localization argument, we see that the above space derivatives $\partial_\omega u$ and $\partial_{\omega\omega}^2 u$ are uniquely determined by (2.9) in $\Lambda(\mathbf{H})$.

Similarly, for each t and $\mathbf{H} \in \mathcal{H}^t$, we may define $\Lambda^t(\mathbf{H})$, $\bar{\Lambda}^t(\mathbf{H})$, $C^0(\Lambda^t(\mathbf{H}))$, $C^0(\bar{\Lambda}^t(\mathbf{H}))$, and $C^{1,2}(\bar{\Lambda}^t(\mathbf{H}))$ in an obvious way.

3 Fully nonlinear path dependent PDEs

In this paper we study the following fully nonlinear parabolic path-dependent partial differential equation (PPDE, for short):

$$\mathcal{L}u(t, \omega) := \{-\partial_t u - G(\cdot, u, \partial_\omega u, \partial_{\omega\omega}^2 u)\}(t, \omega) = 0, \quad 0 \leq t < T, \quad \omega \in \Omega, \quad (3.1)$$

where the generator $G : \Lambda \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}^d \rightarrow \mathbb{R}$ satisfies the following standing assumptions:

Assumption 3.1 The nonlinearity G is nondecreasing in γ and satisfies:

- (i) For fixed (y, z, γ) , $G(\cdot, y, z, \gamma)$ is \mathbb{F} -progressively measurable, and $|G(\cdot, 0, \mathbf{0}, \mathbf{0})| \leq C_0$.
- (ii) G is continuous in (t, ω) under d_∞ .
- (iii) G is uniformly Lipschitz continuous in (y, z, γ) , with a Lipschitz constant L_0 .

Remark 3.2 In the Markovian case, namely $G(t, \omega, \cdot) = g(t, \omega_t, \cdot)$ and $u(t, \omega) = v(t, \omega_t)$, the PPDE (3.1) reduces to the following PDE:

$$\mathbf{L}v(t, x) := \{-\partial_t v(t, x) - g(\cdot, v, D_x v, D_{xx}^2 v)\}(t, x) = 0, \quad t \in [0, T), \quad x \in \mathbb{R}^d. \quad (3.2)$$

Here D_x and D_{xx}^2 denote the standard first and second order derivatives with respect to x . However, slightly different from the PDE literature but consistent with (2.8), ∂_t denotes the right time-derivative. ■

3.1 Classical solutions

Definition 3.3 Let $u \in C^{1,2}(\Lambda)$. We say u is a classical solution (resp. sub-solution, super-solution) of PPDE (3.1) if $\mathcal{L}u(t, \omega) = (\text{resp. } \leq, \geq) 0$ for all $(t, \omega) \in [0, T) \times \Omega$.

It is clear that, in the Markovian setting as in Remark 3.2,

u is a classical solution (resp. sub-solution, super-solution) of PPDE (3.1)

iff v is a classical solution (resp. sub-solution, super-solution) of PDE (3.2).

Example 3.4 Let $d = 1$ and $u(t, \omega) := \mathbb{E}_t^{\mathbb{P}_0} \left[\int_0^T B_t dt \right] = \int_0^t \omega_s ds + (T - t)\omega_t$, $(t, \omega) \in \Lambda$. Then $u \in C^{1,2}(\Lambda)$, and is a classical solution of the path dependent heat equation $-\partial_t u - \frac{1}{2} \partial_{\omega\omega}^2 u = 0$ with final condition $u(T, \omega) = \int_0^T \omega_t dt$. ■

Example 3.5 Let $d = 1$ and $u(t, \omega) := \mathbb{E}_t^{\mathbb{P}_0} \left[\overline{(\omega \otimes_t B^t)_T} \right]$, $(t, \omega) \in \Lambda$ with the notation of Example 2.8. Then one can easily check that $u(t, \omega) = v(t, \omega_t, \overline{\omega}_t)$, where v is a deterministic function defined by:

$$\begin{aligned} v(t, x, y) &:= \mathbb{E}^{\mathbb{P}_0} \left[y \vee (x + \overline{B}_T^t) \right] = x + \sqrt{T-t} \psi \left(\frac{y-x}{\sqrt{T-t}} \right), \quad x \leq y \\ \psi(z) &:= \mathbb{E}^{\mathbb{P}_0} \left[z \vee \overline{B}_1 \right] = \mathbb{E}^{\mathbb{P}_0} \left[z \vee |B_1| \right] = z[2\Phi(z) - 1] + \frac{2}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}, \quad z \geq 0, \end{aligned} \quad (3.3)$$

and Φ denotes the cdf of standard normal. We note that v is smooth when $t < T$, and $\partial_y v(t, x, x) = 0$. Since the support of $d\overline{B}_t$ is in $\{B_t = \overline{B}_t\}$, it follows that $\partial_y v(t, B_t, \overline{B}_t) d\overline{B}_t = 0$. This implies that

$$du(t, \omega) = dv(t, \omega_t, \overline{\omega}_t) = \partial_t v dt + Dv dB_t + \frac{1}{2} D^2 v d\langle B \rangle_t.$$

It is clear that $\partial_t u(t, \omega) = \partial_t v(t, \omega_t, \overline{\omega}_t)$. Then by (2.9) we see that $\partial_\omega u(t, \omega) = Dv(t, \omega_t, \overline{\omega}_t)$ and $\partial_{\omega\omega}^2 u(t, \omega) = D^2 v(t, \omega_t, \overline{\omega}_t)$. Thus $u \in C^{1,2}(\Lambda)$.

Finally, it is straightforwardly to check that u is a classical solution to the path dependent heat equation $-\partial_t u - \frac{1}{2} \partial_{\omega\omega}^2 u = 0$ with final condition $u(T, \omega) = \overline{B}_T$. ■

Remark 3.6 We shall remark that, unlike the standard heat equation which always has classical solution on $[0, T)$, a path dependent heat equation may not have a classical solution on $[0, T)$. One simple example can be the heat equation with terminal condition $u(T, \omega) = B_{t_0}(\omega)$ for some $0 < t_0 < T$. Then clearly $u(t, \omega) = B_{t \wedge t_0}(\omega)$, and thus $\partial_\omega u(t, \omega) = \mathbf{1}_{[0, t_0]}(t)$ is discontinuous. Following Proposition 4.5 below and our accompanying paper [10], and weakening the boundedness assumption as pointed out in Remark 3.8 below, one can easily see u is the unique viscosity solution. We note that Peng and Wang [23] provided some sufficient conditions for the existence of classical solutions of semi-linear PPDEs. ■

3.2 Definition of viscosity solutions

We next introduce our notion of viscosity solutions. Recall the nonlinear Snell envelope notation (2.7). For any $L > 0$, $(t, \omega) \in \Lambda$ with $t < T$, and \mathbb{F} -adapted process u , define

$$\begin{aligned}\underline{\mathcal{A}}^L u(t, \omega) &:= \left\{ \varphi \in C_b^{1,2}(\Lambda^t) : (\varphi - u^{t,\omega})_t(\mathbf{0}) = \underline{\mathcal{S}}_t^L[(\varphi - u^{t,\omega})_{\cdot \wedge H}] \text{ for some } H \in \mathcal{H}^t \right\}, \\ \overline{\mathcal{A}}^L u(t, \omega) &:= \left\{ \varphi \in C_b^{1,2}(\Lambda^t) : (\varphi - u^{t,\omega})_t(\mathbf{0}) = \overline{\mathcal{S}}_t^L[(\varphi - u^{t,\omega})_{\cdot \wedge H}] \text{ for some } H \in \mathcal{H}^t \right\}.\end{aligned}\quad (3.4)$$

Definition 3.7 (i) Let $L > 0$. We say $u \in \underline{\mathcal{U}}$ (resp. $\overline{\mathcal{U}}$) is a viscosity L -subsolution (resp. L -supersolution) of PPDE (3.1) if, for any $(t, \omega) \in [0, T) \times \Omega$ and any $\varphi \in \underline{\mathcal{A}}^L u(t, \omega)$ (resp. $\varphi \in \overline{\mathcal{A}}^L u(t, \omega)$):

$$\{ -\partial_t \varphi - G^{t,\omega}(\cdot, u, \partial_\omega \varphi, \partial_{\omega\omega}^2 \varphi) \}(t, \mathbf{0}) \leq \quad (\text{resp. } \geq) \quad 0.$$

(ii) $u \in \underline{\mathcal{U}}$ (resp. $\overline{\mathcal{U}}$) is a viscosity subsolution (resp. supersolution) of PPDE (3.1) if u is viscosity L -subsolution (resp. L -supersolution) of PPDE (3.1) for some $L > 0$.

(iii) $u \in UC_b(\Lambda)$ is viscosity solution of PPDE (3.1) if it is viscosity sub- and supersolution.

Remark 3.8 For technical simplification, in this paper and the accompanying one [10], we consider only bounded viscosity solutions. By some more involved estimates one can easily extend our theory to viscosity solutions satisfying certain growth conditions. We shall leave this for future research, however, in some examples below we may consider unbounded viscosity solutions as well. \blacksquare

We next provide an intuitive justification of our Definition 3.7 which shows how the above nonlinear optimal stopping problems $\underline{\mathcal{S}}$ and $\overline{\mathcal{S}}$ appear naturally.

Let $u \in C^{1,2}(\Lambda)$ be a classical supersolution of PPDE (3.1), $t^* < T$, $\omega^* \in \Omega$, and $\varphi \in C^{1,2}(\Lambda^{t_0})$. Then:

$$\begin{aligned}0 &\leq \{ -\partial_t u - G(\cdot, u, \partial_\omega u, \partial_{\omega\omega}^2 u) \}(t^*, \omega^*) \\ &= \{ -\partial_t \varphi - G(\cdot, u, \partial_\omega \varphi, \partial_{\omega\omega}^2 \varphi) \}(t^*, \omega^*) + R(t^*, \omega^*)\end{aligned}\quad (3.5)$$

where $R(t, \omega) = \partial_t(\varphi - u) + \hat{\alpha} \cdot \partial_\omega(\varphi - u) + \frac{1}{2} \hat{\beta}^2 : \partial_{\omega\omega}^2(\varphi - u)$, $\hat{\alpha} := G_z(t^*, \omega^*, u(t^*, \omega^*), \hat{z}, \hat{\gamma})$, and $\hat{\beta} := (2G_\gamma(t^*, \omega^*, u(t^*, \omega^*), \hat{z}, \hat{\gamma}))^{1/2}$ are constant drift and diffusion coefficients, and $(\hat{z}, \hat{\gamma})$ are some convex combination of $(\partial_\omega u, \partial_{\omega\omega}^2 u)(t^*, \omega^*)$ and $(\partial_\omega \varphi, \partial_{\omega\omega}^2 \varphi)(t^*, \omega^*)$.

The question is how to choose the test process φ so as to deduce from (3.5) that $0 \leq \{ -\partial_t \varphi - G(\cdot, u, \partial_\omega \varphi, \partial_{\omega\omega}^2 \varphi) \}(t^*, \omega^*)$. Our crucial observation is that

$$d(\varphi - u^{t^*, \omega^*})(t, \omega) = R(t, \omega)dt + \partial_\omega(\varphi - u^{t^*, \omega^*})(t, \omega) \hat{\beta} d\hat{W}_t, \quad \hat{\mathbb{P}} - \text{a.s.}$$

where \hat{W} is a Brownian motion under the probability measure $\hat{\mathbb{P}} \in \mathcal{P}_{L_0}$ defined by the pair $(\hat{\alpha}, \hat{\beta})$, and L_0 is the Lipschitz constant of the nonlinearity G . Therefore, in order to conclude from (3.5) that $R(t^*, \omega^*) \leq 0$, we have to choose the test process φ so that the difference $(\varphi - u^{t^*, \omega^*})$ has a nonpositive $\hat{\mathbb{P}}$ -drift locally at the right hand-side of t^* . This essentially means that $(\varphi - u^{t^*, \omega^*})$ is a $\hat{\mathbb{P}}$ -supermartingale on some right-neighborhood $[t^*, H]$ of t^* , and therefore $(\varphi - u^{t^*, \omega^*})_{t^*} \geq \mathbb{E}^{\hat{\mathbb{P}}}[(\varphi - u^{t^*, \omega^*})_{\tau \wedge H}]$ for any stopping time τ . Since the probability measure $\hat{\mathbb{P}}$ is imposed by the above calculation, we must choose the test process φ so that $(\varphi - u^{t^*, \omega^*})_{t^*} \geq \bar{\mathcal{E}}^L_{t^*}[(\varphi - u^{t^*, \omega^*})_{\tau \wedge H}]$ for all stopping time τ . Finally, since $\tau = t^*$ is a legitimate stopping rule, we arrive at

$$(\varphi - u^{t^*, \omega^*})_{t^*} = \bar{\mathcal{S}}^L_{t^*}[(\varphi - u^{t^*, \omega^*})_{\tau \wedge H}],$$

which corresponds exactly to our definition of $\bar{\mathcal{A}}^L u(t^*, \omega^*)$.

Conversely, if the pair $((t^*, \omega^*), \varphi)$ satisfies the last equality, then it follows that $\varphi - u^{t^*, \omega^*}$ is an $\bar{\mathcal{E}}^L$ -supermartingale near t^* , by the Snell envelope characterization of Theorem 2.4. By the right-continuity, this implies that $R(t^*, \omega^*) \leq 0$. Hence our definition of the set of test processes $\bar{\mathcal{A}}^L u(t^*, \omega^*)$ is essentially necessary and sufficient for the inequality $R(t^*, \omega^*) \leq 0$.

Remark 3.9 From the last intuitive justification of our definition, we see that for a semi-linear path-dependent PDE, $\hat{\beta}$ is a constant matrix. Then, in agreement with our previous paper [8], it is not necessary to vary the coefficient β in the definition of the operator $\bar{\mathcal{E}}^L$.

Similarly, in the context of a linear PPDE, both coefficients $\hat{\alpha}$ and $\hat{\beta}$ are constant, and we may define the sets $\bar{\mathcal{A}}^L u$ and $\underline{\mathcal{A}}^L u$ by means of the linear expectation operator. ■

In the rest of this section we provide several remarks concerning our definition of viscosity solutions. In most places we will comment on the viscosity subsolution only, but obviously similar properties hold for the viscosity supersolution as well.

Remark 3.10 As standard in the literature on viscosity solutions of PDEs:

(i) The viscosity property is a local property in the following sense. For any $(t, \omega) \in [0, T) \times \Omega$ and any $\varepsilon > 0$, define

$$H_\varepsilon^t := \inf \left\{ s > t : |B_s^t| \geq \varepsilon \right\} \wedge (t + \varepsilon) \quad \text{and} \quad H_\varepsilon := H_\varepsilon^0. \quad (3.6)$$

It is clear that $H_\varepsilon^t \in \mathcal{H}^t$. To check the viscosity property of u at (t, ω) , it suffices to know the value of $u^{t, \omega}$ on $[t, H_\varepsilon]$ for an arbitrarily small $\varepsilon > 0$. In particular, since u and φ are locally bounded, there is no integrability issue in (3.4). Moreover, for any $\varphi \in \underline{\mathcal{A}}^L u(t, \omega)$ with corresponding $H \in \mathcal{H}^t$, we have $H_\varepsilon^t \leq H$ when ε is small enough.

- (ii) The fact that u is a viscosity solution does not mean that the PPDE must hold with equality at some (t, ω) and φ in some appropriate set. One has to check viscosity subsolution property and viscosity supersolution property separately.
- (iii) In general $\underline{\mathcal{A}}^L u(t, \omega)$ could be empty. In this case automatically u satisfies the viscosity subsolution property at (t, ω) . ■

Remark 3.11 (i) Consider the Markovian setting in Remark 3.2. One can easily check that u is a viscosity subsolution of PPDE (3.1) in the sense of Definition 3.7 implies that v is a viscosity subsolution of PDE (3.2) in the standard sense, see e.g. [5] or [11]. However, the opposite direction is in general not true. We shall point out though, when the PDE is wellposed, by uniqueness our definition of viscosity solution of PPDE (3.1) is consistent with the viscosity solution of PDE (3.2) in the standard sense.

(ii) The above Definition 3.7 does not reduce to the definition introduced in the semilinear context of [8] either, because we are using a different nonlinear expectation $\bar{\mathcal{E}}^L$ here. It is obvious that any viscosity subsolution in the sense of [8] is also a viscosity subsolution in the sense of this paper, but the opposite direction is in general not true. However, the definitions of viscosity solutions are actually equivalent for semilinear PPDEs, in view of the uniqueness result of our accompanying paper [10]. ■

Remark 3.12 For $0 < L_1 < L_2$, obviously $\mathcal{P}_{L_1}^t \subset \mathcal{P}_{L_2}^t$, $\underline{\mathcal{E}}_t^{L_2} \leq \underline{\mathcal{E}}_t^{L_1}$, and $\underline{\mathcal{A}}^{L_2} u(t, \omega) \subset \underline{\mathcal{A}}^{L_1} u(t, \omega)$. Then one can easily check that a viscosity L_1 -subsolution must be a viscosity L_2 -subsolution. Consequently, u is a viscosity subsolution if and only if

there exists a $L \geq 1$ such that, for all $L' \geq L$, u is a viscosity L' -subsolution.

Remark 3.13 We have some flexibility to choose the set of test functions. All the results in this paper and the accompanying one [10] will still hold true if we replace the $\underline{\mathcal{A}}^L u$ with the $\underline{\mathcal{A}}'^L u$ or $\underline{\mathcal{A}}''^L u$ define below.

(i) The minimum value in the definition of $\underline{\mathcal{A}}^L u$ may be taken to be equal to zero, by replacing φ with $\varphi - \varphi(t, 0) + u(t, \omega)$. That is, we may replace $\underline{\mathcal{A}}^L$ with the following $\underline{\mathcal{A}}'^L$:

$$\underline{\mathcal{A}}'^L u(t, \omega) := \left\{ \varphi \in \underline{\mathcal{A}}^L u(t, \omega) : \varphi(t, \mathbf{0}) = u(t, \omega) \right\}. \quad (3.7)$$

(ii) By the same arguments as in [8] Remark 3.6, we may also replace $\underline{\mathcal{A}}^L$ with the following $\underline{\mathcal{A}}''^L$ with strict extremum in the nonlinear optimal stopping problem:

$$\begin{aligned} \underline{\mathcal{A}}''^L u(t, \omega) &:= \left\{ \varphi \in C^{1,2}(\Lambda^t) : \exists \mathbf{H} \in \mathcal{H} \text{ such that, for all } \tau \in \mathcal{T}^t \text{ with } \tau > t, \right. \\ &\quad \left. (\varphi - u^{t, \omega})_t(\mathbf{0}) = 0 < \underline{\mathcal{E}}_t^L[(\varphi - u^{t, \omega})_{\tau \wedge \mathbf{H}}] \right\}. \end{aligned} \quad (3.8)$$

■

We next report the following result whose proof follows exactly the lines of Remark 3.9 (i) in [8].

Proposition 3.14 *Let Assumption 3.1 hold true, and let u be a viscosity subsolution of PPDE (3.1). For $\lambda \in \mathbb{R}$, the process $\tilde{u}_t := e^{\lambda t} u_t$ is a viscosity subsolution of:*

$$\tilde{\mathcal{L}}\tilde{u} := -\partial_t \tilde{u} - \tilde{G}(t, \omega, \tilde{u}, \partial_\omega \tilde{u}, \partial_{\omega\omega}^2 \tilde{u}) \leq 0, \quad (3.9)$$

where $\tilde{G}(t, \omega, y, z, \gamma) := -\lambda y + e^{\lambda t} G(t, \omega, e^{-\lambda t} y, e^{-\lambda t} z, e^{-\lambda t} \gamma)$.

Remark 3.15 Under Assumption 3.1, we are not able to prove a more general change of variable formula. However, this will be achieved under stronger assumptions, see Proposition 3.5 and Theorem 3.6 of our accompanying paper [10].

3.3 Consistency with classical solutions

Theorem 3.16 *Let Assumption 3.1 hold and $u \in C_b^{1,2}(\Lambda)$. Then u is a classical solution (resp. subsolution, supersolution) of PPDE (3.1) if and only if it is a viscosity solution (resp. subsolution, supersolution).*

Proof We prove the subsolution property only. Assume u is a viscosity L -subsolution. For any (t, ω) , since $u \in C_b^{1,2}(\Lambda)$, we have $u^{t,\omega} \in C_b^{1,2}(\Lambda^t)$ and thus $u^{t,\omega} \in \underline{\mathcal{A}}^L u(t, \omega)$ with $\mathbb{H} := T - t$. By definition of viscosity L -subsolution we see that $\mathcal{L}u(t, \omega) \leq 0$.

On the other hand, assume u is a classical subsolution. If u is not a viscosity subsolution, then it is not a viscosity L_0 -subsolution. Thus there exist $(t, \omega) \in \Lambda$ and $\varphi \in \underline{\mathcal{A}}^{L_0} u(t, \omega)$ as defined in (3.7), such that $2c := \mathcal{L}\varphi(t, \mathbf{0}) > 0$. Without loss of generality, we set $t := 0$ and, by Remark 3.10, let $\mathbb{H} = \mathbb{H}_\varepsilon \in \mathcal{H}$ defined in (3.6) for some small constant $\varepsilon > 0$ be the stopping time used in the definition of $\underline{\mathcal{A}}^{L_0} u(0, \mathbf{0})$. Now let $\mathbb{P} \in \mathcal{P}_{L_0}$ corresponding to some constants $\alpha \in \mathbb{R}^d$ and $\beta \in \mathbb{S}^d$ which will be determined later. Then

$$0 \leq \underline{\mathcal{E}}_0^{L_0} [(\varphi - u)_\mathbb{H}] \leq \mathbb{E}^\mathbb{P} [(\varphi - u)_\mathbb{H}].$$

Applying Itô's formula (2.9) and recalling (2.4) together with the fact that $(\varphi - u)_0 = 0$, we have

$$(\varphi - u)_\mathbb{H} = \int_0^\mathbb{H} \left[\partial_t(\varphi - u)_s + \frac{1}{2} \partial_{\omega\omega}^2(\varphi - u)_s : \beta^2 + \partial_\omega(\varphi - u)_s \cdot \alpha \right] ds + \int_0^\mathbb{H} \partial_\omega(\varphi - u)_s \cdot \beta dW_s^\mathbb{P}.$$

Taking expected values, this leads to

$$0 \leq \mathbb{E}^{\mathbb{P}} \left[\int_0^{\mathbf{H}} \left(\partial_t(\varphi - u)_s + \frac{1}{2} \partial_{\omega\omega}^2(\varphi - u)_s : \beta^2 + \partial_{\omega}(\varphi - u)_s \cdot \alpha \right) ds \right] = \mathbb{E}^{\mathbb{P}} \left[\int_0^{\mathbf{H}} (\tilde{\mathcal{L}}\varphi - \tilde{\mathcal{L}}u)_s ds \right],$$

where $\tilde{\mathcal{L}}\varphi_s := -\mathcal{L}\varphi_s - G(s, \omega, \varphi, \partial_{\omega}\varphi, \partial_{\omega\omega}^2\varphi)_s + \frac{1}{2}(\partial_{\omega\omega}^2\varphi)_s : \beta^2 + (\partial_{\omega}\varphi)_s \cdot \alpha$. Since $\tilde{\mathcal{L}}\varphi$ and $\tilde{\mathcal{L}}u$ are continuous, for ε small enough, we have $|\tilde{\mathcal{L}}\varphi_s - \tilde{\mathcal{L}}\varphi_0| + |\tilde{\mathcal{L}}u_s - \tilde{\mathcal{L}}u_0| \leq \frac{\varepsilon}{2}$ on $[0, \mathbf{H}]$, and then

$$0 \leq \mathbb{E}^{\mathbb{P}} \left[[\tilde{\mathcal{L}}\varphi_0 - \tilde{\mathcal{L}}u_0 + c]\mathbf{H} \right]. \quad (3.10)$$

Note that $\mathcal{L}u_0 \leq 0$, $\mathcal{L}\varphi_0 = 2c$, and $\varphi_0 = u_0$. Then

$$\begin{aligned} \tilde{\mathcal{L}}\varphi_0 - \tilde{\mathcal{L}}u_0 &\leq -2c + \frac{1}{2} \partial_{\omega\omega}^2(\varphi - u)_0 : \beta^2 + \partial_{\omega}(\varphi - u)_0 \cdot \alpha \\ &\quad - [G(0, \mathbf{0}, u_0, \partial_{\omega}\varphi_0, \partial_{\omega\omega}^2\varphi_0) - G(0, \mathbf{0}, u_0, \partial_{\omega}u_0, \partial_{\omega\omega}^2u_0)]. \end{aligned}$$

By Assumption 3.1 (iii), there exist α and β such that $\mathbb{P} \in \mathcal{P}_{L_0}$ and

$$G(\cdot, u, \partial_{\omega}\varphi, \partial_{\omega\omega}^2\varphi)_0 - G(\cdot, u, \partial_{\omega}u, \partial_{\omega\omega}^2u)_0 = \frac{1}{2} \partial_{\omega\omega}^2(\varphi - u)_0 : \beta^2 + \partial_{\omega}(\varphi - u)_0 \cdot \alpha.$$

Then $\tilde{\mathcal{L}}\varphi_0 - \tilde{\mathcal{L}}u_0 \leq -2c$, and (3.10) leads to $0 \leq \mathbb{E}^{\mathbb{P}}[-c\mathbf{H}] < 0$, contradiction. \blacksquare

4 Some Examples with Representation Formula

In this section we provide several examples of viscosity solutions.

4.1 First order PPDEs

Example 4.1 Suppose that $u(t, \omega) = v(\omega_t)$ for all $(t, \omega) \in \Lambda$. Then $\partial_t u = 0$, by the definition of the time-derivative. We now verify that u is a viscosity solution of the equation $-\partial_t u = 0$.

Indeed, for $\varphi \in \overline{\mathcal{A}}^L u(t, \omega)$, it follows from our definition that, for some $\mathbf{H} \in \mathcal{H}^t$:

$$(\varphi - u^{t, \omega})_t(\mathbf{0}) \geq \mathbb{E}^{\mathbb{P}^{0,0}} [(\varphi - u^{t, \omega})_{(t+\delta) \wedge \mathbf{H}}] \quad \text{for all } \delta > 0.$$

Here $\mathbb{P}^{0,0}$ is the probability measure corresponding to $\alpha = \mathbf{0}, \beta = \mathbf{0}$ in (2.4). Notice that under $\mathbb{P}^{0,0}$, the canonical process ω is frozen to its value at time t . Then, for δ small enough,

$$\mathbb{E}^{\mathbb{P}^{0,0}} [(\varphi - u^{t, \omega})_{(t+\delta) \wedge \mathbf{H}}] = \varphi(t + \delta, \mathbf{0}) - v(\omega_t).$$

This implies that $\partial_t \varphi(t, \mathbf{0}) \leq 0$. A similar argument shows that $\partial_t \varphi(t, \mathbf{0}) \geq 0$ for all $\varphi \in \underline{\mathcal{A}}^L u(t, \omega)$. \blacksquare

Example 4.2 Let $d = 1$. In this example we check that $u(t, \omega) := 2\bar{B}_t - B_t$ is a viscosity solution of the first order equation:

$$-\partial_t u - |\partial_\omega u| + 1 = 0. \quad (4.1)$$

By Example 2.8, u is not smooth, so it is a viscosity solution but not a classical solution.

When $\omega_t < \bar{\omega}_t$, it is clear that u is smooth with $\partial_t u(t, \omega) = 0, \partial_\omega u(t, \omega) = -1$ and thus satisfies (4.1). So it suffices to check the viscosity property when $\omega_t = \bar{\omega}_t$. Without loss of generality, we check it at $(t, \omega) = (0, 0)$.

(i) We first check that $\underline{\mathcal{A}}'^L u(0, 0)$, as defined in (3.7), is empty for $L \geq 1$, and thus u is a viscosity subsolution. Indeed, assume $\varphi \in \underline{\mathcal{A}}'^L u(0, 0)$ with corresponding $\mathbf{H} \in \mathcal{H}$. By Remark 3.10 (i), without loss of generality we may assume $\mathbf{H} = \mathbf{H}_\varepsilon$ for some small $\varepsilon > 0$, and thus $\partial_t \varphi, \partial_{\omega\omega} \varphi$ are bounded on $[0, \mathbf{H}]$. Note that $\mathbb{P}_0 \in \mathcal{P}_L$. By definition of $\underline{\mathcal{A}}'^L$ we have, for any $0 < \delta < \varepsilon$,

$$\begin{aligned} 0 &\leq \mathbb{E}^{\mathbb{P}_0}[(\varphi - u)_{\delta \wedge \mathbf{H}}] = \mathbb{E}^{\mathbb{P}_0}\left[\int_0^{\delta \wedge \mathbf{H}} [\partial_t \varphi + \partial_{\omega\omega}^2 \varphi](t, \omega) ds - 2\bar{B}_{\delta \wedge \mathbf{H}}\right] \\ &\leq C\mathbb{E}^{\mathbb{P}_0}[\delta \wedge \mathbf{H}] - 2\mathbb{E}^{\mathbb{P}_0}[\bar{B}_{\delta \wedge \mathbf{H}}] \leq C\delta + C\mathbb{P}_0(\mathbf{H} \leq \delta) - 2\mathbb{E}^{\mathbb{P}_0}[\bar{B}_\delta] + 2\mathbb{E}^{\mathbb{P}_0}[\bar{B}_T \mathbf{1}_{\{\mathbf{H} \leq \delta\}}] \\ &\leq C\delta - 2\sqrt{\delta} + C\mathbb{P}_0(\mathbf{H} \leq \delta) + C\sqrt{\mathbb{P}_0(\mathbf{H} \leq \delta)}. \end{aligned}$$

Note that

$$\mathbb{P}_0(\mathbf{H} \leq \delta) = \mathbb{P}_0(\mathbf{H}_\varepsilon \leq \delta) = \mathbb{P}_0(\|B\|_\delta \geq \varepsilon) \leq \varepsilon^{-4} \mathbb{E}^{\mathbb{P}_0}[\|B\|_\delta^4] \leq C\varepsilon^{-4}\delta^2. \quad (4.2)$$

Then

$$0 \leq C[\delta + \varepsilon^{-4}\delta^2 + \varepsilon^{-2}\delta] - 2\sqrt{\delta}.$$

This leads to a contradiction when δ is small enough. Therefore, $\underline{\mathcal{A}}'^L u(0, \mathbf{0})$ is empty.

(ii) We next check the viscosity supersolution property. Assume to the contrary that $-c := -\partial_t u(0, 0) - |\partial_\omega \varphi(0, 0)| + 1 < 0$ for some $\varphi \in \bar{\mathcal{A}}'^L u(0, 0)$ and $L \geq 1$. Let $\alpha := \text{sgn}(\partial_\omega \varphi(0, 0))$, $\beta := 0$, and \mathbb{P} be determined by (2.4). Then $\mathbb{P} \in \mathcal{P}_L$ and $B_t = \bar{B}_t = t, \mathbf{H}_\varepsilon = \varepsilon$, \mathbb{P} -a.s. By choosing $\mathbf{H} = \mathbf{H}_\varepsilon$ and ε small enough, we may assume $|\partial_t \varphi(t, B) - \partial_t \varphi(0, 0)| + |\partial_\omega \varphi(t, B) - \partial_\omega \varphi(0, 0)| \leq \frac{c}{2}$ for $t \leq \mathbf{H}_\varepsilon$. By the definition of $\bar{\mathcal{A}}'^L u(0, 0)$ we get

$$\begin{aligned} 0 &\geq \mathbb{E}^{\mathbb{P}}[(\varphi - u)_{\mathbf{H}_\varepsilon}] = \mathbb{E}^{\mathbb{P}}\left[\int_0^\varepsilon (\partial_t \varphi + \alpha \partial_\omega \varphi)_t dt - \varepsilon\right] \\ &\geq \mathbb{E}^{\mathbb{P}}\left[\int_0^\varepsilon \left(\partial_t \varphi_0 + \alpha \partial_\omega \varphi_0 - \frac{c}{2}\right) dt\right] - \varepsilon \\ &= \mathbb{E}^{\mathbb{P}}\left[\int_0^\varepsilon \left(\partial_t \varphi_0 + |\partial_\omega \varphi_0| - \frac{c}{2}\right) dt\right] - \varepsilon = \int_0^\varepsilon \left(1 + c - \frac{c}{2}\right) dt - \varepsilon = \frac{1}{2} c\varepsilon > 0. \end{aligned}$$

This is the required contradiction, and thus u is a viscosity supersolution of (4.1). ■

4.2 Semi-linear PPDEs and BSDEs

We now consider the following semi-linear PPDE:

$$-\partial_t u - \frac{1}{2}\sigma^2(t, \omega) : \partial_{\omega\omega}^2 u - F(t, \omega, u, \partial_\omega u \sigma(t, \omega)) = 0, \quad u(T, \omega) = \xi(\omega), \quad (4.3)$$

where $\sigma \geq \mathbf{0}$ and F are \mathbb{F} -progressively measurable and ξ is \mathcal{F}_T -measurable. We note that [8] studied the case $\sigma = I_d$ for simplicity. We shall assume

- Assumption 4.3** (i) σ , $F(t, \omega, 0, \mathbf{0})$, and ξ are bounded by C_0 .
(ii) σ , F are uniformly continuous in (t, ω) and ξ is uniformly continuous in ω , with a common modulus of continuity function ρ_0 .
(iii) $\sigma > \mathbf{0}$ is uniformly Lipschitz continuous in ω , and F is uniformly Lipschitz continuous in (y, z) .

Remark 4.4 The boundedness in Assumption 4.3 (i) is just for simplification, which can be easily weakened to some growth conditions. The assumption $\sigma > \mathbf{0}$ and that F depends on the gradient term through the special form $\partial_\omega u \sigma(t, \omega)$ are mainly needed for the subsequent BSDE representation. As we will see in our accompanying paper [10], one may study the more general PPDE with nonlinearity $F(t, \omega, u, \partial_\omega u)$ depending directly on $\partial_\omega u$. ■

For any $(t, \omega) \in \Lambda$, consider the following decoupled FBSDE on $[t, T]$:

$$\begin{cases} \mathcal{X}_s = \int_t^s \sigma^{t, \omega}(r, \mathcal{X}) dB_r^t, \\ \mathcal{Y}_s = \xi^{t, \omega}(\mathcal{X}) + \int_s^T F^{t, \omega}(r, \mathcal{X}, \mathcal{Y}_r, \mathcal{Z}_r) dr - \int_s^T \mathcal{Z}_r dB_r^t, \end{cases} \quad \mathbb{P}_0^t - \text{a.s.} \quad (4.4)$$

Under Assumption 4.3, clearly FBSDE (4.4) is wellposed and we denote its solution as $(\mathcal{X}^{t, \omega}, \mathcal{Y}^{t, \omega}, \mathcal{Z}^{t, \omega})$. Alternatively, we may consider the BSDE in weak formulation:

$$Y_s^{t, \omega} = \xi^{t, \omega}(B^t) + \int_s^T F^{t, \omega}(r, B^t, Y_r^{t, \omega}, Z_r^{t, \omega}) dr - \int_s^T Z_r^{t, \omega} (\sigma^{t, \omega}(r, B^t))^{-1} dB_r^t, \quad \mathbb{P}^{t, \omega} - \text{a.s.} \quad (4.5)$$

where $\mathbb{P}^{t, \omega} := \mathbb{P}_0^t \circ (\mathcal{X}^{t, \omega})^{-1}$. Then, for any fixed (t, ω) ,

$$\mathcal{Y}_t^{t, \omega} = Y_t^{t, \omega} \quad \text{and} \quad \text{is a constant due to the Blumenthal zero-one law.}$$

Proposition 4.5 Under Assumption 4.3, $u(t, \omega) := Y_t^{t, \omega} = \mathcal{Y}_t^{t, \omega}$ is a viscosity solution of PPDE (4.3).

Proof We proceed in two steps.

Step 1. In Step 2 below, we will show that $u \in UC_b(\Lambda)$ and satisfies the dynamic programming principle: for any $(t, \omega) \in \Lambda$ and $\tau \in \mathcal{T}^t$,

$$Y_s^{t,\omega} = u^{t,\omega}(\tau, B^t) + \int_s^\tau F^{t,\omega}(r, B^t, Y_r^{t,\omega}, Z_r^{t,\omega})dr - \int_s^\tau Z_r^{t,\omega}(\sigma^{t,\omega}(r, B^t))^{-1}dB_r^t, \quad \mathbb{P}^{t,\omega}\text{-a.s.} \quad (4.6)$$

Let L be a constant such that $\sigma \leq \sqrt{2L}I_d$ and $\sqrt{\frac{L}{2}}$ is a Lipschitz constant of F in (y, z) . We now show that u is an L -viscosity solution. Without loss of generality, we prove only the viscosity subsolution property at $(t, \omega) = (0, \mathbf{0})$. For notational simplicity we omit the superscript 0,0 in the rest of this proof. Assume to the contrary that,

$$c := -\left\{\partial_t \varphi + \frac{1}{2}\sigma^2 : \partial_{\omega\omega}^2 \varphi + F(\cdot, u, \partial_\omega \varphi \sigma)\right\}(0, \mathbf{0}) > 0 \quad \text{for some } \varphi \in \underline{\mathcal{A}}^L u(0, \mathbf{0}).$$

Let $\mathbf{H} \in \mathcal{H}$ be the hitting time corresponding to φ in (3.4), and by Remark 3.10 (i), without loss of generality we may assume $\mathbf{H} = \mathbf{H}_\varepsilon$ for some small $\varepsilon > 0$. Since $\varphi \in C_b^{1,2}(\Lambda)$, by the uniform continuity of u , σ , and f , we may assume ε is small enough such that

$$-\left\{\partial_t \varphi + \frac{1}{2}\sigma^2 : \partial_{\omega\omega}^2 \varphi + F(\cdot, u, \partial_\omega \varphi \sigma)\right\}(t, \omega) \geq \frac{c}{2} > 0, \quad t \in [0, \mathbf{H}].$$

Notice that $d\langle B \rangle_t = \sigma^2(t, B.)dt$, \mathbb{P} -a.s. Using the dynamic programming principle (4.6), and applying Itô's formula on φ , we have:

$$\begin{aligned} & (\varphi - u)_0 - (\varphi - u)_\mathbf{H} \\ &= -\int_0^\mathbf{H} \left(\partial_t \varphi + \frac{1}{2}\sigma^2 : \partial_{\omega\omega}^2 \varphi + F(\cdot, u_s, Z_s) \right)(s, B.)ds - \int_0^\mathbf{H} (\partial_\omega \varphi - Z_s \sigma^{-1})(s, B.)dB_s \\ &\geq \int_0^\mathbf{H} \left(\frac{c}{2} + F(\cdot, u, \partial_\omega \varphi \sigma) - F(\cdot, u_s, Z_s) \right)(s, B.)ds - \int_0^\mathbf{H} (\partial_\omega \varphi - Z_s \sigma^{-1})(s, B.)dB_s \\ &= \int_0^\mathbf{H} \left[\frac{c}{2} + (\partial_\omega \varphi \sigma - Z_s) \cdot \alpha_s \right](s, B.)ds - \int_0^\mathbf{H} (\partial_\omega \varphi - Z_s \sigma^{-1})(s, B.)dB_s \\ &= \frac{c}{2}\mathbf{H} - \int_0^\mathbf{H} (\partial_\omega \varphi \sigma - Z_s)(s, B.)[\sigma^{-1}(s, B.)dB_s - \alpha_s ds], \quad \mathbb{P}\text{-a.s.} \end{aligned}$$

where $|\alpha| \leq \sqrt{\frac{L}{2}}$. Notice that $\sigma^{-1}dB_t$ is a \mathbb{P} -Brownian motion. Applying Girsanov Theorem one sees immediately that there exists $\tilde{\mathbb{P}} \in \mathcal{P}_L$ equivalent to \mathbb{P} such that $\sigma^{-1}dB_t - \alpha_t dt$ is a $\tilde{\mathbb{P}}$ -Brownian motion. Then the above inequality holds $\tilde{\mathbb{P}}$ -a.s., and by the definition of $\underline{\mathcal{A}}^L u$:

$$0 \geq (\varphi - u)_0 - \mathbb{E}^{\tilde{\mathbb{P}}}[(\varphi - u)_\mathbf{H}] \geq \frac{c}{2}\mathbb{E}^{\tilde{\mathbb{P}}}[\mathbf{H}] > 0,$$

which is the required contradiction.

Step 2. We now show the dynamic programming principle together with the following

regularity of u : for any $t \leq t'$ and any ω, ω' ,

$$|u(t, \omega)| \leq C \quad \text{and} \quad |u(t, \omega) - u(t', \omega')| \leq C\bar{\rho}_0\left(d_\infty((t, \omega), (t', \omega'))\right), \quad (4.7)$$

where

$$\bar{\rho}_0(\delta) := \delta^{\frac{1}{3}} + \rho_0(\delta^{\frac{1}{3}}). \quad (4.8)$$

Indeed, by standard arguments it is clear that

$$\begin{aligned} \mathbb{E}^{\mathbb{P}_0^t} \left[\|\mathcal{X}^{t, \omega}\|_T^2 + \|\mathcal{Y}^{t, \omega}\|_T^2 + \int_t^T |\mathcal{Z}_s^{t, \omega}|^2 ds \right] &\leq C; \\ \mathbb{E}^{\mathbb{P}_0^t} \left[\|\mathcal{X}^{t, \omega} - \mathcal{X}^{t, \omega'}\|_T^2 + \|\mathcal{Y}^{t, \omega} - \mathcal{Y}^{t, \omega'}\|_T^2 + \int_t^T |\mathcal{Z}_s^{t, \omega} - \mathcal{Z}_s^{t, \omega'}|^2 ds \right] &\leq C\rho_0(\|\omega - \omega'\|_t)^2; \\ \mathbb{E}^{\mathbb{P}^{t, \omega}} \left[\|B^t\|_T^2 + \|Y^{t, \omega}\|_T^2 + \int_t^T |Z_s^{t, \omega}|^2 ds \right] &\leq C \end{aligned}$$

In particular, this implies that

$$|u(t, \omega)| \leq C \quad \text{and} \quad |u(t, \omega) - u(t, \omega')| \leq C\rho_0(\|\omega - \omega'\|_t). \quad (4.9)$$

Given the above regularity, by standard arguments in BSDE theory, we have the following dynamic programming principle: for any $t < t' \leq T$,

$$Y_s^{t, \omega} = u^{t, \omega}(t', B^t) + \int_s^{t'} F^{t, \omega}(r, B^t, Y_r^{t, \omega}, Z_r^{t, \omega}) dr - \int_s^{t'} Z_r^{t, \omega} (\sigma^{t, \omega}(r, B^t))^{-1} dB_r^t, \quad \mathbb{P}^{t, \omega}\text{-a.s.} \quad (4.10)$$

In particular, $Y_s^{t, \omega} = u^{t, \omega}(s, B^t)$.

Denote $\delta := d_\infty((t, \omega), (t', \omega'))$. If $\delta \geq \frac{1}{2}$, clearly

$$|u_t - u_{t'}|(\omega) \leq C \leq C\bar{\rho}_0(\delta).$$

In the rest of this proof, we assume $\delta \leq \frac{1}{2}$. Then

$$\begin{aligned} |u_t - u_{t'}|(\omega) &= \left| \mathbb{E}^{\mathbb{P}^{t, \omega}} \left[Y_t^{t, \omega} - Y_{t'}^{t, \omega} + u^{t, \omega}(t', B^t) - u(t', \omega) \right] \right| \\ &= \left| \mathbb{E}^{\mathbb{P}^{t, \omega}} \left[\int_t^{t'} F^{t, \omega}(r, B^t, Y_r^{t, \omega}, Z_r^{t, \omega}) dr + u^{t, \omega}(t', B^t) - u(t', \omega) \right] \right| \\ &\leq \mathbb{E}^{\mathbb{P}^{t, \omega}} \left[\int_t^{t'} |F^{t, \omega}(r, B^t, Y_r^{t, \omega}, Z_r^{t, \omega})| dr + C\rho_0\left(\delta + \|B^t\|_{t'}\right) \right], \quad (4.11) \end{aligned}$$

Notice that

$$\begin{aligned} \mathbb{E}^{\mathbb{P}^{t, \omega}} \left[\int_t^{t'} |F^{t, \omega}(r, B^t, Y_r^{t, \omega}, Z_r^{t, \omega})| dr \right] &\leq C\mathbb{E}^{\mathbb{P}^{t, \omega}} \left[\int_t^{t'} (1 + |Y_r^{t, \omega}| + |Z_r^{t, \omega}|) dr \right] \\ &\leq C\sqrt{\delta} \left(\mathbb{E}^{\mathbb{P}^{t, \omega}} \left[\int_t^{t'} (1 + |Y_r^{t, \omega}|^2 + |Z_r^{t, \omega}|^2) dr \right] \right)^{\frac{1}{2}} \\ &\leq C\sqrt{\delta}. \end{aligned}$$

As for the second term, we estimate, assuming without loss of generality $\rho_0 \leq C_0$,

$$\begin{aligned}
\mathbb{E}^{\mathbb{P}^{t,\omega}} \left[\rho_0 \left(\delta + \|B^t\|_{t'} \right) \right] &\leq \rho_0(\delta^{\frac{1}{3}}) + C_0 \mathbb{P}^{t,\omega} \left[\|B^t\|_{t'} \geq \delta^{\frac{1}{3}} - \delta \right] \\
&\leq \rho_0(\delta^{\frac{1}{3}}) + \frac{C_0}{(\delta^{\frac{1}{3}} - \delta)^2} \mathbb{E}^{\mathbb{P}^{t,\omega}} \left[\|B^t\|_{t'}^2 \right] \\
&\leq \rho_0(\delta^{\frac{1}{3}}) + \frac{C}{(\delta^{\frac{1}{3}} - \delta)^2} \mathbb{E}^{\mathbb{P}^{t,\omega}} \left[\int_t^{t'} |\sigma^{t,\omega}(s, B^t)|^2 ds \right] \\
&\leq \rho_0(\delta^{\frac{1}{3}}) + \frac{C\delta}{(\delta^{\frac{1}{3}} - \delta)^2} \leq \rho_0(\delta^{\frac{1}{3}}) + C\delta^{\frac{1}{3}} \leq C\bar{\rho}_0(\delta).
\end{aligned}$$

Plugging the last estimates in (4.11), and combining with (4.9), we obtain (4.7). Moreover, given the regularity in t , we may extend the dynamic programming principle (4.10) to stopping times, proving (4.6). \blacksquare

Remark 4.6 For FBSDE (4.4) with $(t, \omega) = (0, \mathbf{0})$, it holds that $\mathcal{Y}_t := u(t, \mathcal{X})$. This extends the well known nonlinear Feynman-Kac formula in [18] to the path dependent case. \blacksquare

4.3 Path dependent HJB equations and 2BSDEs

Let \mathbb{K} be an arbitrary set in some measurable space. We now consider the following path dependent HJB equation:

$$\begin{aligned}
-\partial_t u - G(t, \omega, u, \partial_\omega u, \partial_\omega^2 u) &= 0, \quad u(T, \omega) = \xi(\omega); \\
\text{where } G(t, \omega, y, z, \gamma) &:= \sup_{k \in \mathbb{K}} \left[\frac{1}{2} \sigma^2(t, \omega, k) : \gamma + F(t, \omega, y, z\sigma(t, \omega, k), k) \right],
\end{aligned} \tag{4.12}$$

where σ and F are \mathbb{F} -progressively measurable in all variables. We shall assume

Assumption 4.7 (i) σ , $F(t, \omega, 0, \mathbf{0}, k)$, and ξ are bounded by C_0 .
(ii) σ , F are uniformly continuous in (t, ω) and ξ is uniformly continuous in ω , with a common modulus of continuity function ρ_0 .
(iii) $\sigma > \mathbf{0}$ is uniformly Lipschitz continuous in ω , and F is uniformly Lipschitz continuous in (y, z) .

For each t , let \mathcal{K}^t denote the set of \mathbb{F}^t -progressively measurable \mathbb{K} -valued processes on Λ^t . For any $(t, \omega) \in \Lambda$ and $k \in \mathcal{K}^t$, let $\mathcal{X}^{t,\omega,k}$ denote the solution to the following SDE:

$$\mathcal{X}_s = \int_t^s \sigma^{t,\omega}(r, \mathcal{X}, k_r) dB_r^t, \quad t \leq s \leq T, \quad \mathbb{P}_0^t\text{-a.s.}$$

Denote $\mathbb{P}^{t,\omega,k} := \mathbb{P}_0^t \circ (\mathcal{X}^{t,\omega,k})^{-1}$. Since $\sigma > \mathbf{0}$, as discussed in [25] $\mathcal{X}^{t,\omega,k}$ and B^t induce the same \mathbb{P}_0^t -augmented filtration, and thus there exists $\tilde{k} \in \mathcal{K}^t$ such that $\tilde{k}(\mathcal{X}^{t,\omega,k}) = k$, \mathbb{P}_0^t -a.s. Let $(Y^{t,\omega,k}, Z^{t,\omega,k})$ denote the solution to the following BSDE on $[t, T]$:

$$Y_s = \xi^{t,\omega}(B^t) + \int_s^T F^{t,\omega}(r, B^t, Y_r, Z_r, \tilde{k}_r) dr - \int_s^T Z_r (\sigma^{t,\omega}(r, B^t, \tilde{k}_r))^{-1} dB_r^t, \quad \mathbb{P}^{t,\omega,k}\text{-a.s.}$$

We now consider the stochastic control problem:

$$u(t, \omega) := \sup_{k \in \mathcal{K}^t} Y_t^{t,\omega,k}, \quad (t, \omega) \in \Lambda.$$

We observe that this process u was considered by Nutz [15], in the stochastic control context, and shown to be the solution of a second order BSDE. The next result shows that our notion of viscosity solution is also suitable for this stochastic control problem.

Proposition 4.8 *Under Assumption 4.7, u is a viscosity solution of PPDE (4.12).*

Proof By Proposition 3.14, without loss of generality we assume

$$G, \text{ hence } F, \text{ is increasing in } y. \quad (4.13)$$

Following similar arguments as in Proposition 4.5, we may prove that:

$$|u(t, \omega)| \leq C, \quad |u(t, \omega) - u(t', \omega')| \leq C\bar{\rho}_0(d_\infty((t, \omega), (t', \omega'))), \quad \text{for any } (t, \omega), (t', \omega') \in \Lambda,$$

and that the following dynamic programming principle holds

$$u(t, \omega) = \sup_{k \in \mathcal{K}^t} \mathcal{Y}_t^{t,\omega,k}(\tau, u^{t,\omega}(\tau, \cdot)), \quad \text{for any } (t, \omega) \in \Lambda, \tau \in \mathcal{T}^t, \quad (4.14)$$

where, for any \mathcal{F}_τ^t -measurable random variable η , $(\mathcal{Y}, \mathcal{Z}) := (\mathcal{Y}^{t,\omega,k}(\tau, \eta), \mathcal{Z}^{t,\omega,k}(\tau, \eta))$ solves the following BSDE on $[t, \tau]$:

$$\mathcal{Y}_s = \eta(B^t) + \int_s^\tau F^{t,\omega}(r, B^t, \mathcal{Y}_r, \mathcal{Z}_r, \tilde{k}_r) dr - \int_s^\tau \mathcal{Z}_r (\sigma^{t,\omega}(r, B^t, \tilde{k}_r))^{-1} dB_r^t, \quad \mathbb{P}^{t,\omega,k} - \text{a.s.}$$

See e.g. [25] or [19]. We now prove the viscosity property, for the same L as in Proposition 4.5. Again we shall only prove it at $(t, \omega) = (0, \mathbf{0})$ and we will omit the superscript $^{0,\mathbf{0}}$. However, since in this case u is defined through a supremum, we need to prove the viscosity subsolution property and supersolution property separately.

Viscosity L -subsolution property. Assume to the contrary that,

$$c := -\{\partial_t \varphi + G(\cdot, u, \partial_\omega \varphi, \partial_{\omega\omega}^2 \varphi)\}(0, \mathbf{0}) > 0 \quad \text{for some } \varphi \in \underline{\mathcal{A}}^L u(0, \mathbf{0}).$$

As in Proposition 4.5, let $\mathbf{H} = \mathbf{H}_\varepsilon \in \mathcal{H}$ be the hitting time corresponding to φ in (3.4), and since $\varphi \in C_b^{1,2}(\Lambda)$ and u , σ , and F are uniformly continuous, we assume ε is small enough such that

$$-\{\partial_t \varphi + G(\cdot, u, \partial_\omega \varphi, \partial_{\omega\omega}^2 \varphi)\}(t, \omega) \geq \frac{c}{2} > 0, \quad t \in [0, \mathbf{H}].$$

By the definition of G , this implies that, for any $t \in [0, \mathbf{H}]$ and $k \in \mathbb{K}$,

$$-\{\partial_t \varphi + \frac{1}{2} \sigma^2(t, \omega, k) : \partial_{\omega\omega}^2 \varphi + F(t, \omega, u, \partial_\omega \varphi \sigma(\cdot, k), k)\}(t, \omega) \geq \frac{c}{2} > 0.$$

Now for any $k \in \mathcal{K}$, notice that $d\langle B \rangle_t = \sigma^2(t, B, \tilde{k}_t)dt$, \mathbb{P}^k -a.s. Denote $(\mathcal{Y}^k, \mathcal{Z}^k) := (\mathcal{Y}^k(\mathbf{H}, u(\mathbf{H}, \cdot)), \mathcal{Z}^k(\mathbf{H}, u(\mathbf{H}, \cdot)))$. Applying Itô's formula on φ , we see that for any $\delta > 0$:

$$\begin{aligned} (\varphi - \mathcal{Y}^k)_0 - (\varphi - u)_{\mathbf{H} \wedge \delta} &= (\varphi - \mathcal{Y}^k)_0 - (\varphi - \mathcal{Y}^k)_{\mathbf{H} \wedge \delta} \\ &= - \int_0^{\mathbf{H} \wedge \delta} \left[\partial_t \varphi + \frac{1}{2} \sigma^2 : \partial_{\omega\omega}^2 \varphi + F(\cdot, \mathcal{Y}_s^k, \mathcal{Z}_s^k) \right] (s, B, \tilde{k}_s) ds \\ &\quad - \int_0^{\mathbf{H} \wedge \delta} (\partial_\omega \varphi - Z_s \sigma^{-1})(s, B, \tilde{k}_s) dB_s \\ &\geq \int_0^{\mathbf{H} \wedge \delta} \left[\frac{c}{2} + F(\cdot, u, \partial_\omega \varphi \sigma) - F(\cdot, \mathcal{Y}_s^k, \mathcal{Z}_s^k) \right] (s, B, \tilde{k}_s) ds \\ &\quad - \int_0^{\mathbf{H} \wedge \delta} (\partial_\omega \varphi - \mathcal{Z}_s^k \sigma^{-1})(s, B, \tilde{k}_s) dB_s. \end{aligned}$$

Note that $\mathcal{Y}_s^k \leq u(s, B)$. Then by (4.13) we have

$$\begin{aligned} &[\varphi - \mathcal{Y}^k]_0 - [\varphi - u]_{\mathbf{H} \wedge \delta} \\ &\geq \int_0^{\mathbf{H} \wedge \delta} \left[\frac{c}{2} + F(\cdot, u, \partial_\omega \varphi \sigma) - F(\cdot, u, \mathcal{Z}_s^k) \right] (s, B, \tilde{k}_s) ds \\ &\quad - \int_0^{\mathbf{H} \wedge \delta} (\partial_\omega \varphi - \mathcal{Z}_s^k \sigma^{-1})(s, B, \tilde{k}_s) dB_s \\ &= \int_0^{\mathbf{H} \wedge \delta} \left[\frac{c}{2} + (\partial_\omega \varphi \sigma - \mathcal{Z}_s^k) \alpha_s \right] (s, B, \tilde{k}_s) ds - \int_0^{\mathbf{H} \wedge \delta} (\partial_\omega \varphi - \mathcal{Z}_s^k \sigma^{-1})(s, B, \tilde{k}_s) dB_s \\ &= \frac{c}{2} (\mathbf{H} \wedge \delta) - \int_0^{\mathbf{H} \wedge \delta} (\partial_\omega \varphi - \mathcal{Z}_s^k \sigma^{-1})(s, B, \tilde{k}_s) (dB_s - \sigma \alpha_s ds), \quad \mathbb{P}^k\text{-a.s.} \end{aligned}$$

where $|\alpha| \leq \sqrt{\frac{L}{2}}$ and λ is bounded. As in Proposition 4.5, we may define $\tilde{\mathbb{P}}^k \in \mathcal{P}_L$ equivalent to \mathbb{P} such that $dB_t - \sigma \alpha_t dt$ is a $\tilde{\mathbb{P}}^k$ -martingale. Then the above inequality holds $\tilde{\mathbb{P}}^k$ -a.s., and by the definition of $\underline{\mathcal{A}}^L u$, we have

$$u_0 - \mathcal{Y}_0^k \geq u_0 - \mathcal{Y}_0^k + (\varphi - u)_0 - \mathbb{E}^{\tilde{\mathbb{P}}^k} [(\varphi - u)_{\mathbf{H} \wedge \delta}] \geq \frac{c}{2} \mathbb{E}^{\tilde{\mathbb{P}}^k} [\mathbf{H} \wedge \delta] \geq \frac{c}{2} \delta - C \mathbb{P}^k[\mathbf{H} \leq \delta].$$

Following the same arguments as in (4.2), we have $\tilde{\mathbb{P}}^k[\mathbf{H} \leq \delta] \leq C\varepsilon^{-4}\delta^2$, where the constant C is independent of k . Then, for δ small enough,

$$u_0 - \mathcal{Y}_0^k \geq \frac{c}{2}\delta - C\varepsilon^{-4}\delta^2 \geq \frac{c\delta}{4} > 0.$$

This implies that $u_0 - \sup_{k \in \mathcal{K}} \mathcal{Y}_0^k \geq \frac{c\delta}{4} > 0$, which is in contradiction with (4.14).

Viscosity L -supersolution property. Assume to the contrary that,

$$c := \left\{ \partial_t \varphi + G(\cdot, u, \partial_\omega \varphi, \partial_{\omega\omega}^2 \varphi) \right\}(0, \mathbf{0}) > 0 \quad \text{for some } \varphi \in \overline{\mathcal{A}}^L u(0, \mathbf{0}).$$

By the definition of F , there exists $k_0 \in \mathbb{K}$ such that

$$\left\{ \partial_t \varphi + \frac{1}{2} \sigma^2(\cdot, k_0) : \partial_{\omega\omega}^2 \varphi + F(\cdot, u, \partial_\omega \varphi \sigma(\cdot, k_0), k_0) \right\}(0, \mathbf{0}) \geq \frac{c}{2} > 0$$

Again, let $\mathbf{H} = \mathbf{H}_\varepsilon \in \mathcal{H}$ be the hitting time corresponding to φ in (3.4), and since $\varphi \in C_b^{1,2}(\Lambda(\mathbf{H}))$ and u, σ , and G are uniformly continuous, we assume ε is small enough so that

$$\left\{ \partial_t \varphi + \frac{1}{2} \sigma^2(\cdot, k_0) : \partial_{\omega\omega}^2 \varphi + F(\cdot, u, \partial_\omega \varphi \sigma(\cdot, k_0), k_0) \right\}(t, \omega) \geq \frac{c}{3} > 0, \quad t \in [0, \mathbf{H}].$$

Consider the constant process $k := k_0 \in \mathcal{K}$. It is clear that the corresponding $\tilde{k} = k_0$. Follow similar arguments as in the subsolution property, we arrive at the following contradiction:

$$u_0 - \mathcal{Y}_0^k \leq -\frac{c}{3} \mathbb{E}^{\tilde{\mathbb{P}}^k}[\mathbf{H}] < 0.$$

The proof is complete now. ■

Example 4.9 Assume $\mathbb{K} := \{k \in \mathbb{S}^d : \underline{\sigma} \leq k \leq \overline{\sigma}\}$, where $\mathbf{0} < \underline{\sigma} < \overline{\sigma}$ are constant matrices. Set $\sigma(t, \omega, k) := k$. Then $Y_t(\omega) = u(t, \omega)$ is the solution to the following second order BSDE, as introduced by Soner, Touzi and Zhang [26]:

$$Y_t = \xi(B_\cdot) + \int_t^T F(s, B_\cdot, Y_s, Z_s, \hat{a}_s^{\frac{1}{2}}) ds - \int_t^T Z_s(\hat{a}_s)^{-\frac{1}{2}} dB_s - dK_t, \mathcal{P}\text{-q.s.} \quad (4.15)$$

where $\mathcal{P} := \{\mathbb{P} \in \mathcal{P}_\infty : \alpha^\mathbb{P} = 0, \beta^\mathbb{P} \in \mathbb{K}\}$, \hat{a} is the universal process such that $d\langle B \rangle_t = \hat{a}_t dt$, \mathcal{P} -q.s. and K is an increasing process satisfying certain minimum condition. A closely related notion is the G-BSDE of [12]. In particular, when $F = 0$, then $u_t = \mathbb{E}_t^G[g(B)]$ is the Peng [20] conditional G -expectation. ■

Remark 4.10 By using the zero-sum game, we may also obtain a representation formula for the viscosity solution of the following path dependent Bellman-Isaacs equation:

$$-\partial_t u - G(t, \omega, u, \partial_\omega u, \partial_{\omega\omega}^2 u) = 0, \quad u(T, \omega) = \xi(\omega) \quad (4.16)$$

where

$$\begin{aligned} G(t, \omega, y, z, \gamma) &:= \sup_{k_1 \in \mathbb{K}_1} \inf_{k_2 \in \mathbb{K}_2} \left[\frac{1}{2} \sigma^2(t, k_1, k_2) : \gamma + F(t, \omega, y, z\sigma(t, k_1, k_2), k_1, k_2) \right] \\ &= \inf_{k_2 \in \mathbb{K}_2} \sup_{k_1 \in \mathbb{K}_1} \left[\frac{1}{2} \sigma^2(t, k_1, k_2) : \gamma + F(t, \omega, y, z\sigma(t, k_1, k_2), k_1, k_2) \right]. \end{aligned}$$

See Pham and Zhang [24]. ■

5 Stability and Partial Comparison

5.1 Stability

The main result of this section is the following extension of Theorem 4.1 in [8], with a proof following the same line of argument. However the present fully nonlinear context makes a crucial use of Theorem 2.4.

Theorem 5.1 *Let $(G^\varepsilon, \varepsilon > 0)$ be a family of coefficients satisfying Assumptions 3.1 uniformly, and G^ε converge uniformly towards a coefficient G as $\varepsilon \rightarrow 0$. For some $L > 0$, let u^ε be a bounded viscosity L -subsolution (resp. L -supersolution) of PPDE (3.1) with coefficients G^ε , for all $\varepsilon > 0$. Assume further that u^ε converges to some $u \in \underline{\mathcal{U}}$ (resp $u \in \overline{\mathcal{U}}$), uniformly in Λ . Then u is a viscosity L -subsolution (resp. L -supersolution) of PPDE (3.1) with coefficient G .*

Proof We shall prove only the viscosity supersolution property by contradiction. By Remark 3.13, without loss of generality we assume there exists $\varphi \in \underline{\mathcal{A}}''^L u(0, \mathbf{0})$ such that $-c := \mathcal{L}\varphi(0, \mathbf{0}) < 0$, where $\underline{\mathcal{A}}''^L u(0, \mathbf{0})$ is defined in (3.8).

By Remark 3.10 (i), let $\mathbf{H} = \mathbf{H}_\delta \in \mathcal{H}$ be the hitting time corresponding to φ in (3.8). Since $\varphi \in C^{1,2}$ and $\mathcal{L}\varphi(0, \mathbf{0}) = -c < 0$, by choosing $\delta > 0$ small enough, we have

$$\mathcal{L}\varphi(t, \omega) \leq -\frac{c}{2} \quad \text{for all } (t, \omega) \in \bar{\Lambda}(\mathbf{H}). \quad (5.1)$$

Denote $X^0 := \varphi - u$ and $X^\varepsilon := \varphi - u^\varepsilon$. By (3.8), $\bar{\mathcal{E}}_0^L[X_{\mathbf{H}}^0] < 0 = X_0^0$. Since u^ε converges towards u uniformly, and $u_{\mathbf{H}}^\varepsilon \leq u_{\mathbf{H}-}^\varepsilon$ by the definition of $\overline{\mathcal{U}}$, this implies that

$$\bar{\mathcal{E}}_0^L[X_{\mathbf{H}-}^\varepsilon] \leq \bar{\mathcal{E}}_0^L[X_{\mathbf{H}}^\varepsilon] < X_0^\varepsilon \quad \text{for sufficiently small } \varepsilon > 0. \quad (5.2)$$

Apply Theorem 2.4 to $(X^\varepsilon, \mathbf{H})$ and introduce the corresponding process \widehat{X}^ε , its nonlinear Snell envelope Y^ε , and the first hitting time τ^* that $Y^\varepsilon = \widehat{X}^\varepsilon$. Then, by Theorem 2.4:

$$\bar{\mathcal{E}}_0^L[\widehat{X}_{\mathbf{H}}^\varepsilon] < X_0^\varepsilon \leq Y_0^\varepsilon = \bar{\mathcal{E}}_0^L[Y_{\tau^*}^\varepsilon] = \bar{\mathcal{E}}_0^L[\widehat{X}_{\tau^*}^\varepsilon].$$

Thus there exists $\omega^* \in \Omega$ such that $t^* := \tau^*(\omega^*) < \mathsf{H}(\omega^*)$, and it follows from the $\overline{\mathcal{E}}^L$ -supermartingale property of Theorem 2.4 that

$$X_{t^*}^\varepsilon(\omega^*) = Y_{t^*}^\varepsilon(\omega^*) \geq \overline{\mathcal{E}}_{t^*}^L[Y_\tau^{\varepsilon, t^*, \omega^*}] \geq \overline{\mathcal{E}}_{t^*}^L[X_\tau^{\varepsilon, t^*, \omega^*}] \quad \text{for all } \tau \in \mathcal{T}^{t^*}, \tau < \mathsf{H}^{t^*, \omega^*}.$$

Note that the set O , corresponding to H , is open, then there exists $\tilde{\mathsf{H}} \in \mathcal{H}^{t^*}$ such that $\tilde{\mathsf{H}} < \mathsf{H}^{t^*, \omega^*}$. Thus

$$X_{t^*}^\varepsilon(\omega^*) \geq \overline{\mathcal{E}}_{t^*}^L[X_{\tau \wedge \tilde{\mathsf{H}}}^{\varepsilon, t^*, \omega^*}] \quad \text{for all } \tau \in \mathcal{T}^{t^*}.$$

This implies that $\varphi^{t^*, \omega^*} \in \overline{\mathcal{A}}^L u^\varepsilon(t^*, \omega^*)$. Since u^ε is a viscosity L -supersolution of PPDE (3.1) with coefficients G^ε , by (5.1) we have

$$\begin{aligned} 0 &\leq \{ -\partial_t \varphi - G^\varepsilon(\cdot, u^\varepsilon, \partial_\omega \varphi, \partial_{\omega\omega}^2 \varphi) \}(t^*, \omega^*) \\ &\leq -\frac{\varepsilon}{2} + \{ G(\cdot, \varphi, \partial_\omega \varphi, \partial_{\omega\omega}^2 \varphi) - G^\varepsilon(\cdot, u^\varepsilon, \partial_\omega \varphi, \partial_{\omega\omega}^2 \varphi) \}(t^*, \omega^*). \end{aligned}$$

Note that

$$|\varphi - u^\varepsilon|(t^*, \omega^*) \leq |\varphi(t^*, \omega^*) - \varphi(t, \mathbf{0})| + |u(t^*, \omega^*) - u(t, \mathbf{0})| + |u - u^\varepsilon|(t^*, \omega^*).$$

Then, by sending $\varepsilon \rightarrow 0$ and choosing δ small enough, we obtain $0 \leq -\frac{\varepsilon}{3}$, contradiction. \blacksquare

Remark 5.2 Similar to Theorem 4.1 in [8], we need the same L in the proof of Theorem 5.1. If u^ε is only a viscosity subsolution of PPDE (3.1) with coefficient G^ε , but with possibly different L_ε , we are not able to show that u is a viscosity subsolution of PPDE (3.1) with coefficient G . \blacksquare

5.2 Partial comparison of viscosity solutions

In this section, we prove a partial comparison principle, i.e. a comparison result of a viscosity super- (resp. sub-) solution and a classical sub- (resp. super-) solution. The proof is also crucially based on Theorem 2.4. Moreover, this result is a first key step for our comparison principle in the accompanying paper [10].

Proposition 5.3 *Let Assumption 3.1 hold true. Let $u^1 \in \underline{\mathcal{U}}$ be a viscosity subsolution and $u^2 \in \overline{\mathcal{U}}$ a viscosity supersolution of PPDE (3.1). If $u^1(T, \cdot) \leq u^2(T, \cdot)$ and one of u^1 and u^2 is in $C^{1,2}(\Lambda)$, then $u^1 \leq u^2$ on Λ .*

Proof We shall only prove $u_0^1 \leq u_0^2$. The inequality for general t can be proved similarly. Without loss of generality, we assume u^1 is a viscosity L -subsolution and $u^2 \in C^{1,2}(\Lambda)$ is viscosity L -supersolution. By Proposition 3.14, we may assume that

$$G \text{ is nonincreasing in } y. \quad (5.3)$$

Assume to the contrary that $c := \frac{1}{2T}[u_0^1 - u_0^2] > 0$. Denote

$$X_t := (u^1 - u^2)_t^+ + ct, \quad \widehat{X}_t := X_t \mathbf{1}_{\{t < T\}} + X_T \mathbf{1}_{\{t = T\}}; \quad Y_t(\omega) := \sup_{\tau \in \mathcal{T}^t} \overline{\mathcal{E}}_t^L[\widehat{X}_\tau^{t,\omega}].$$

Since $u^1 \in \underline{\mathcal{U}}$ is bounded from above, and $u^2 \in \overline{\mathcal{U}} \cap C^{1,2}(\Lambda)$ is bounded from below, it follows that X is a bounded process in $\underline{\mathcal{U}}$. Let τ^* be the first time that $Y = \widehat{X}$. Note that $u_T^1 \leq u_T^2$, and $X \in \underline{\mathcal{U}}$ has positive jumps. Then it follows from Theorem 2.4 that

$$\overline{\mathcal{E}}_0^L[\widehat{X}_T] \leq \overline{\mathcal{E}}_0^L[X_T] = cT < 2cT = X_0 = \widehat{X}_0 \leq Y_0 = \overline{\mathcal{E}}_0^L[Y_{\tau^*}] = \overline{\mathcal{E}}_0^L[\widehat{X}_{\tau^*}].$$

This implies that $t^* := \tau^*(\omega^*) < T$ for some $\omega^* \in \Omega$. Note that

$$(u^1 - u^2)^+(t^*, \omega^*) + ct^* = \widehat{X}_{t^*}(\omega^*) = Y_{t^*}(\omega^*) \geq \overline{\mathcal{E}}_{t^*}^L[X_{T-}^{t^*, \omega^*}] \geq Tc > 0.$$

Then $(u^1 - u^2)(t^*, \omega^*) > 0$. Since $u^1 - u^2 \in \underline{\mathcal{U}}^*$, there exists $H \in \mathcal{H}^{t^*}$ such that $H < T$ and $(u^1 - u^2)_t^{t^*, \omega^*} > 0$ for all $t \in [t^*, H]$, and thus $\widehat{X}_t^{t^*, \omega^*} = X_t^{t^*, \omega^*} = (u^1 - u^2)_t^{t^*, \omega^*} + ct$ for all $t \in [t^*, H]$.

Now observe that $\varphi(t, \omega) := (u^2)^{t^*, \omega^*}(t, \omega) - ct \in C^{1,2}(\Lambda^{t^*})$, a consequence of our assumption $u^2 \in C^{1,2}(\Lambda)$. Moreover, for any $\tau \in \mathcal{T}^{t^*}$, it follows from the $\overline{\mathcal{E}}^L$ -supermartingale property of the nonlinear Snell envelope Y that

$$((u^1)^{t^*, \omega^*} - \varphi)_{t^*} = Y_{t^*}(\omega^*) \geq \overline{\mathcal{E}}_{t^*}^L[Y_{\tau \wedge H}^{t^*, \omega^*}] \geq \overline{\mathcal{E}}_{t^*}^L[X_{\tau \wedge H}^{t^*, \omega^*}] \geq \overline{\mathcal{E}}_{t^*}^L[((u^1)^{t^*, \omega^*} - \varphi)_{\tau \wedge H}].$$

By the arbitrariness of $\tau \in \mathcal{T}^{t^*}$, and the fact that $\underline{\mathcal{E}}^L[\cdot] = -\overline{\mathcal{E}}^L[-\cdot]$, this proves that $\varphi \in \underline{\mathcal{A}}^L u^1(t^*, \omega^*)$, and by the viscosity L -subsolution property of u^1 :

$$\begin{aligned} 0 &\geq \{ -\partial_t \varphi - G(\cdot, u^1, \partial_\omega \varphi, \partial_{\omega\omega}^2 \varphi) \}(t^*, \omega^*) \\ &= c - \{ \partial_t u^2 + G(\cdot, u^1, \partial_\omega u^2, \partial_{\omega\omega}^2 u^2) \}(t^*, \omega^*) \\ &\geq c - \{ \partial_t u^2 + G(\cdot, u^2, \partial_\omega u^2, \partial_{\omega\omega}^2 u^2) \}(t^*, \omega^*), \end{aligned}$$

where the last inequality follows from (5.3). Since $c > 0$, this is in contradiction with the supersolution property of u^2 . ■

As a direct consequence of the above partial comparison, we have

Proposition 5.4 *Let Assumption 3.1 hold true. If PPDE (3.1) has a classical solution u , then u is the unique viscosity solution of PPDE (3.1) with terminal condition $u(T, \cdot)$.*

In our accompanying paper [10], we shall prove the uniqueness of viscosity solutions without assuming the existence of classical solutions.

6 Viscosity Solutions of Backward Stochastic PDE

In this section, we show that our PPDE includes Backward SPDE as a special case. We remark that such BSPDEs arise naturally in many applications, see e.g. [14] and [16]. Consider the following BSPDE with adapted solution (u, q) :

$$u(t, \omega, x) = \xi(\omega, x) + \int_t^T F(s, \omega, x, u, Du, D^2u, q, Dq)ds - \int_t^T q(s, \omega, x)dB_s, \mathbb{P}_0\text{-a.s.} \quad (6.1)$$

where $x \in \mathbb{R}^{d'}$, and D, D^2 denote the partial gradient and Hessian with respect to the x -variable. Assume $u \in C^{1,2,2}$, where the derivatives in x are in standard sense and the smoothness in (t, ω) is in the sense of Definition 2.6. Fix x and apply Itô's formula we have

$$du(t, \omega, x) = \left(\partial_t u + \frac{1}{2} \partial_{\omega\omega}^2 u \right)(t, \omega, x)dt + \partial_\omega u(t, \omega, x)dB_t, \quad \mathbb{P}_0\text{-a.s.}$$

Comparing this with (6.1) we obtain

$$q(t, \omega, x) = \partial_\omega u(t, \omega, x), \quad \left(\partial_t u + \frac{1}{2} \partial_{\omega\omega}^2 u \right)(t, \omega, x) + F(t, \omega, x, u, Du, D^2u, q, Dq) = 0.$$

This leads to a mixed PPDE:

$$\widehat{\mathcal{L}}u(t, \omega, x) = 0, \quad u(T, \omega, x) = h(\omega, x), \quad x \in \mathbb{R}, \quad (6.2)$$

where, for $\varphi \in C^{1,2,2}$,

$$\widehat{\mathcal{L}}\varphi := -\partial_t \varphi - \frac{1}{2} \partial_{\omega\omega}^2 \varphi - F(\cdot, \varphi, D\varphi, D^2\varphi, \partial_\omega \varphi, D\partial_\omega \varphi). \quad (6.3)$$

To incorporate the mixed PPDE (6.2) into our framework, we enlarge the space of paths to $\hat{\Omega} := \Omega \times \{\omega' \in C^0([0, T], \mathbb{R}^{d'}) : \omega'_0 = 0\}$, where d' is the dimension of x . Denote $\hat{\Lambda} := [0, T] \times \hat{\Omega}$, and for all $x \in \mathbb{R}^{d'}$:

$$\begin{aligned} \hat{G}^x(t, \hat{\omega}, y, z, \gamma) &:= \frac{1}{2} \gamma_{22} + F(t, \omega, x + \omega'_t, y, z_1, \gamma_{11}, z_2, \gamma_{12})(t, \omega, x + \omega'_t), \\ \hat{\xi}^x(\hat{\omega}) &:= \xi(\omega, x + \omega'_T), \end{aligned}$$

so that $\hat{G}^x(t, \cdot)$ and $\hat{\xi}^x$ depend on $\hat{\omega} = (\omega, \omega')$ only through the pair (ω, ω'_t) and (ω, ω'_T) , respectively.

Definition 6.1 We say u is a viscosity solution (resp. supersolution, subsolution) of BSPDE (6.1) if, for any fixed x , the process $\hat{u}^x(t, \hat{\omega}) := u(t, \omega, x + \omega'_t)$, $t \in [0, T]$, $\hat{\omega} = (\omega, \omega') \in \hat{\Omega}$, is a viscosity solution (resp. supersolution, subsolution) of the PPDE:

$$-\partial_t \hat{u}^x(t, \hat{\omega}) - \hat{G}^x(t, \hat{\omega}, \hat{u}^x, \partial_{\hat{\omega}} \hat{u}^x, \partial_{\hat{\omega}\hat{\omega}} \hat{u}^x) = 0, \quad \text{on } \hat{\Lambda}, \quad \text{and} \quad \hat{u}^x(T, \hat{\omega}) = \hat{\xi}^x(\hat{\omega}).$$

Remark 6.2 In the same manner we may also transform the following Stochastic PDE into a (forward) PPDE:

$$u(t, x) = u_0(x) + \int_0^t F(s, x, u, u_x, u_{xx}) ds + \int_0^t q(s, x, u, u_x) dB_s. \quad (6.4)$$

Due to its forward nature, the definition of viscosity solutions will be quite different. However, the approach which will be specified in next section and in [10] still works in this case. See Buckdahn, Ma, and Zhang [2]. ■

7 A revisit of semi-linear PPDE

In [8], we proved the comparison principle for semilinear PPDE (4.3), in the case $\sigma = I_d$. One important argument there is the Bank-Baum approximation in [1], which unfortunately does not seem to be extendable to the fully nonlinear case. In this section we provide an alternative proof of the comparison principle for semilinear PPDE (4.3). This approach works in fully nonlinear case as well, but with much more involved technicalities, see our accompanying paper [10].

In order to focus on the main idea and simplify the presentation, we restrict to the case $\sigma = I_d$. That is, we shall consider the following PPDE:

$$\mathcal{L}u(t, \omega) := -\partial_t u - \frac{1}{2} I_d : \partial_{\omega\omega}^2 u - F(t, \omega, u, \partial_{\omega} u) = 0, \quad u(T, \omega) = \xi(\omega). \quad (7.1)$$

We first give an alternative definition for viscosity solutions of semilinear PPDE (7.1). We remark that the key point in (3.4) and Definition 3.7 is that the class \mathcal{P}_L covers all the probability measures induced by the linearization of generator F . In the semilinear case, since the diffusion term σ is already fixed, we shall only consider the drift uncertainty

induced by the linearization of generator F , as we did in [8]. To be precise, define

$$\begin{aligned}
M_T^{t,\alpha} &:= \exp \left(\int_t^T \alpha_s dB_s^t - \frac{1}{2} \int_t^T |\alpha_s|^2 ds \right); \\
\mathcal{P}_L^t &:= \left\{ \mathbb{P}(\cdot) := \int M_T^{t,\alpha} d\mathbb{P}_0^t : \alpha \text{ is } \mathbb{F}^t\text{-prog. meas. } \mathbb{R}^d\text{-valued and } |\alpha| \leq L \right\}; \\
\bar{\mathcal{E}}_t^L[\xi] &:= \sup_{\mathbb{P} \in \mathcal{P}_L^t} \mathbb{E}^{\mathbb{P}}[\xi], \quad \underline{\mathcal{E}}_t^L[\xi] := \inf_{\mathbb{P} \in \mathcal{P}_L^t} \mathbb{E}^{\mathbb{P}}[\xi]; \\
\underline{\mathcal{A}}^L u(t, \omega) &:= \left\{ \varphi \in C_b^{1,2}(\Lambda^t) : (\varphi - u^{t,\omega})_t(\mathbf{0}) = \inf_{\tau \in \mathcal{T}^t} \underline{\mathcal{E}}_t^L[(\varphi - u^{t,\omega})_{\cdot \wedge \mathbf{H}}] \text{ for some } \mathbf{H} \in \mathcal{T}_+^t \right\}, \\
\overline{\mathcal{A}}^L u(t, \omega) &:= \left\{ \varphi \in C_b^{1,2}(\Lambda^t) : (\varphi - u^{t,\omega})_t(\mathbf{0}) = \sup_{\tau \in \mathcal{T}^t} \bar{\mathcal{E}}_t^L[(\varphi - u^{t,\omega})_{\cdot \wedge \mathbf{H}}] \text{ for some } \mathbf{H} \in \mathcal{T}_+^t \right\}.
\end{aligned}$$

Here $\mathcal{T}_+^t := \{\tau \in \mathcal{T}^t : \tau > t\}$. We note that in this case we don't need to assume $\mathbf{H} \in \mathcal{H}^t$ since one can easily solve the optimal stopping problem by using the Reflected BSDE theory.

Definition 7.1 (i) Let $L > 0$. We say $u \in \underline{\mathcal{U}}$ (resp. $\overline{\mathcal{U}}$) is a viscosity L -subsolution (resp. L -supersolution) of semilinear PPDE (7.1) if, for any $(t, \omega) \in [0, T) \times \Omega$ and any $\varphi \in \underline{\mathcal{A}}^L u(t, \omega)$ (resp. $\varphi \in \overline{\mathcal{A}}^L u(t, \omega)$):

$$\left\{ -\partial_t \varphi - \frac{1}{2} \text{tr}(\partial_{\omega\omega}^2 \varphi) - F^{t,\omega}(\cdot, u, \partial_\omega \varphi) \right\}(t, \mathbf{0}) \leq \quad (\text{resp. } \geq) \quad 0.$$

(ii) $u \in \underline{\mathcal{U}}$ (resp. $\overline{\mathcal{U}}$) is a viscosity subsolution (resp. supersolution) of PPDE (7.1) if u is viscosity L -subsolution (resp. L -supersolution) of PPDE (7.1) for some $L > 0$.

(iii) $u \in UC_b(\Lambda)$ is viscosity solution of PPDE (7.1) if it is viscosity sub- and supersolution.

Under Assumption 4.3, following almost the same arguments and after obvious modification if necessary, one can easily check that Theorems 3.16, 5.1, and Proposition 4.5 still hold. Moreover, we may improve the partial comparison principle of Proposition 5.3 as follows. First, we extend the space $C^{1,2}(\Lambda)$:

Definition 7.2 Let $t \in [0, T]$, $u : \Lambda^t \rightarrow \mathbb{R}$. We say $u \in \bar{C}^{1,2}(\Lambda^t)$ if there exist an increasing sequence of \mathbb{F}^t -stopping times $t = \mathbf{H}_0 \leq \mathbf{H}_1 \leq \dots \leq T$ such that,

- (i) $\mathbf{H}_i < \mathbf{H}_{i+1}$ whenever $\mathbf{H}_i < T$, and for all $\omega \in \Omega^t$, the set $\{i : \mathbf{H}_i(\omega) < T\}$ is finite;
- (ii) For each $i \geq 0$ and $\omega \in \Omega^t$, $\mathbf{H}_{i+1}^{(\omega), \omega} \in \mathcal{T}^{\mathbf{H}_i(\omega)}$ and $u^{\mathbf{H}_i(\omega), \omega} \in C_b^{1,2}(\Lambda^{\mathbf{H}_i(\omega)}(\mathbf{H}_{i+1}^{(\omega), \omega}))$;
- (iii) $u \in C_b^0(\bar{\Lambda}^t)$, and

$$\mathbb{E}^{\mathbb{P}_0^t} \left[\int_0^T \left[|\partial_t u + \frac{1}{2} \partial_{\omega\omega}^2 u|^2 + |\partial_\omega u|^2 \right] (t, B) dt \right] < \infty. \quad (7.2)$$

We note that the space $\bar{C}^{1,2}(\Lambda^t)$ here is slightly different from that in [8], and in [10] we shall modify it slightly further for technical reasons. Following the arguments in [8],

one may easily check that the partial comparison principle Proposition 5.3 still holds if one requires either u^1 or u^2 is only in $\overline{C}^{1,2}(\Lambda)$, instead of $C^{1,2}(\Lambda)$.

We now turn to comparison and uniqueness. First, define

$$\overline{u}(t, \omega) := \inf \{ \psi(t, \mathbf{0}) : \psi \in \overline{\mathcal{D}}(t, \omega) \}, \quad \underline{u}(t, \omega) := \sup \{ \psi(t, \mathbf{0}) : \psi \in \underline{\mathcal{D}}(t, \omega) \}, \quad (7.3)$$

where, for the \mathcal{L} in (7.1),

$$\begin{aligned} \overline{\mathcal{D}}(t, \omega) &:= \left\{ \psi \in \overline{C}^{1,2}(\Lambda^t) : (\mathcal{L}\psi)^{t,\omega} \geq 0 \text{ on } [t, T) \times \Omega^t \text{ and } \psi_T \geq \xi^{t,\omega} \right\}, \\ \underline{\mathcal{D}}(t, \omega) &:= \left\{ \psi \in \overline{C}^{1,2}(\Lambda^t) : (\mathcal{L}\psi)^{t,\omega} \leq 0 \text{ on } [t, T) \times \Omega^t \text{ and } \psi_T \leq \xi^{t,\omega} \right\}. \end{aligned} \quad (7.4)$$

By the improved version of the partial comparison result of Proposition 5.3, it is clear that

$$\underline{u} \leq \overline{u}. \quad (7.5)$$

A crucial step for our proof is to show that equality holds in the last inequality.

Proposition 7.3 *Under Assumption 4.3 with $\sigma = I_d$, we have $\overline{u} = \underline{u}$.*

We then have the following wellposedness result.

Theorem 7.4 *Let Assumption 4.3 hold and $\sigma = I_d$.*

- (i) *Let u^1 be a bounded viscosity subsolution and u^2 a bounded viscosity supersolution of semilinear PPDE (7.1), in the sense of Definition 7.1, with $u^1(T, \cdot) \leq \xi \leq u^2(T, \cdot)$. Then $u^1 \leq u^2$ on Λ .*
- (ii) *The semilinear PPDE (7.1) with terminal condition ξ has a unique viscosity solution $u \in UC_b(\Lambda)$, in the sense of Definition 7.1.*

Proof First by the partial comparison principle, we have $u^1 \leq \overline{u}$ and $\underline{u} \leq u^2$. Then Proposition 7.3 implies $u^1 \leq u^2$ immediately, which implies further the uniqueness of viscosity solution. Finally by Proposition 4.5 we have the existence. \blacksquare

Proof of Proposition 7.3. Without loss of generality, we shall only prove $\overline{u}(0, \mathbf{0}) \leq \underline{u}(0, \mathbf{0})$. In light of Proposition 3.14, we may assume without loss of generality that

$$F \text{ is nonincreasing in } y. \quad (7.6)$$

For any $\varepsilon > 0$, we denote

$$\begin{aligned} O_\varepsilon &:= \{x \in \mathbb{R}^d : |x| < \varepsilon\}, \quad \overline{O}_\varepsilon := \{x \in \mathbb{R}^d : |x| \leq \varepsilon\}, \quad \partial O_\varepsilon := \{x \in \mathbb{R}^d : |x| = \varepsilon\}; \\ \mathcal{O}_t^\varepsilon &:= [t, T) \times O_\varepsilon, \quad \overline{\mathcal{O}}_t^\varepsilon := [t, T] \times \overline{O}_\varepsilon, \quad \partial \mathcal{O}_t^\varepsilon := ([t, T) \times \partial O_\varepsilon) \cup (\{T\} \times O_\varepsilon). \end{aligned}$$

Let $t_0 = 0$, $x_0 = 0$, $(t_i)_{i \geq 1}$ an increasing sequence in $(0, T]$ with $t_i = T$ when i is large enough, and $(x_i)_{i \geq 1}$ a sequence in $\partial\mathcal{O}_\varepsilon$. Set $\pi := (t_i, x_i)_{i \geq 0}$ and $\pi_n := (t_i, x_i)_{0 \leq i \leq n}$. Given π_n and $(t, x) \in \mathcal{O}_{t_n}^\varepsilon$, define

$$H_0^{t,x,\varepsilon} := \{s \geq t : |B_s^t + x| = \varepsilon\} \wedge T, \quad H_{i+1}^{t,x,\varepsilon} := T \wedge \inf \{s \geq H_i^{t,x,\varepsilon} : |B_s^t - B_{H_i^{t,x,\varepsilon}}^t| = \varepsilon\}, \quad i \geq 0.$$

For $t \in (t_n, T]$, let $\hat{B}^{\varepsilon, \pi_n, t, x}(\omega)$ denote the linear interpolation of $(t_i, \sum_{j=0}^i x_j)_{0 \leq i \leq n}$ and $(H_i^{t,x,\varepsilon}(\omega), \sum_{j=0}^n x_j + x + B_{H_i^{t,x,\varepsilon}}^t(\omega))_{i \geq 0}$. Define

$$\theta_n^\varepsilon(\pi_n; (t, x)) := \mathcal{Y}_t^{\varepsilon, \pi_n, t, x}$$

where, denoting $H_{-1}^{t,x,\varepsilon} := t$ and omitting the superscripts ε, π_n, t, x ,

$$\mathcal{Y}_s = \xi(\hat{B}) + \int_s^T F\left(s, \sum_{i \geq -1} \hat{B}_{\cdot \wedge H_i^{t,x,\varepsilon}} \mathbf{1}_{[H_i^{t,x,\varepsilon}, H_{i+1}^{t,x,\varepsilon})}, \mathcal{Y}_s, \mathcal{Z}_s\right) ds - \int_s^T \mathcal{Z}_s dB_s, \quad \mathbb{P}_0^t\text{-a.s.}$$

It is straightforward to check that for all n and π_n , the deterministic function $\theta_n^\varepsilon := \theta_n^\varepsilon(\pi_n; \cdot)$ satisfies the following standard PDE on $\mathcal{O}_{t_n}^\varepsilon$:

$$\begin{aligned} -\partial_t \theta_n^\varepsilon - \frac{1}{2} I_d : D^2 \theta_n^\varepsilon - F(s, \hat{\omega}^{\pi_n}, \theta_n^\varepsilon, D\theta_n^\varepsilon) &= 0 \text{ in } \mathcal{O}_{t_n}^\varepsilon, \\ \theta_n^\varepsilon(\pi_n; t, x) &= \theta_{n+1}^\varepsilon(\pi_n, (t, x); t, \mathbf{0}) \text{ on } \partial\mathcal{O}_{t_n}^\varepsilon, \end{aligned} \quad (7.7)$$

where $\hat{\omega}^{\pi_n} := \hat{B}_{\cdot \wedge t_n}^{\varepsilon, \pi_n, t, x}$ is deterministic, and $\theta_n^\varepsilon(\pi_n; T, x) = \xi(\hat{\omega}^{\pi_n})$ when $t_n = T$. In particular, by the PDE theory we see that $\theta_n^\varepsilon \in C^{1,2}(\mathcal{O}_{t_n}^\varepsilon)$.

We now let $H_i^\varepsilon := H_i^{0, \mathbf{0}, \varepsilon}$, and \hat{B}^ε the linear interpolation of $(H_i^\varepsilon, B_{H_i^\varepsilon})_{i \geq 0}$. Define

$$\psi^\varepsilon(t, \omega) := \sum_{n=0}^{\infty} \theta_n^\varepsilon((H_i^\varepsilon, B_{H_i^\varepsilon})_{0 \leq i \leq n}; t, B_t - B_{H_n^\varepsilon}) \mathbf{1}_{[H_n^\varepsilon, H_{n+1}^\varepsilon)}.$$

One can check straightforwardly that ψ^ε is bounded, $\psi^\varepsilon(T, \omega) = \xi(\hat{B}^\varepsilon)$, and

$$-\partial_t \psi^\varepsilon - \frac{1}{2} I_d : \partial_{\omega\omega}^2 \psi^\varepsilon - F\left(s, \sum_{i \geq -1} \hat{B}_{\cdot \wedge H_i^\varepsilon} \mathbf{1}_{[H_i^\varepsilon, H_{i+1}^\varepsilon)}, \psi^\varepsilon, \partial_\omega \psi^\varepsilon\right) = 0. \quad (7.8)$$

Then $Y := \psi^\varepsilon, Z := \partial_\omega \psi^\varepsilon$ satisfy the following BSDE:

$$Y_t = \xi(\hat{B}^\varepsilon) + \int_t^T F\left(s, \sum_{i \geq -1} \hat{B}_{\cdot \wedge H_i^\varepsilon} \mathbf{1}_{[H_i^\varepsilon, H_{i+1}^\varepsilon)}, Y_s, Z_s\right) ds - \int_t^T Z_s dB_s, \quad \mathbb{P}_0\text{-a.s.}$$

This implies (7.2) and then $\psi^\varepsilon \in \overline{C}^{1,2}(\Lambda)$ with the corresponding stopping times H_n^ε . Notice that $\|\hat{B}^\varepsilon - B\|_T \leq \varepsilon$. Then

$$\|\xi(\hat{B}^\varepsilon) - \xi(B)\| \leq \rho_0(\varepsilon), \quad \left| F\left(s, \sum_{i=-1}^{\infty} \hat{B}_{\cdot \wedge H_i^\varepsilon} \mathbf{1}_{[H_i^\varepsilon, H_{i+1}^\varepsilon)}, y, z\right) - F(s, B, y, z) \right| \leq \rho_0(\varepsilon). \quad (7.9)$$

Set

$$\overline{\psi}^\varepsilon := \psi^\varepsilon + \rho_0(\varepsilon)[1 + T - t], \quad \underline{\psi}^\varepsilon := \psi^\varepsilon - \rho_0(\varepsilon)[1 + T - t].$$

Then

$$\overline{\psi}^\varepsilon \geq \psi^\varepsilon, \quad \overline{\psi}^\varepsilon \in \overline{C}^{1,2}(\Lambda), \quad \overline{\psi}^\varepsilon(T, \omega) \geq \psi^\varepsilon(T, \omega) + \rho_0(\varepsilon) = \xi(\hat{B}^\varepsilon) + \rho_0(\varepsilon) \geq \xi(B),$$

and, by (7.6), (7.9), and (7.8)

$$\begin{aligned} & -\partial_t \overline{\psi}^\varepsilon - \frac{1}{2} Id : \partial_{\omega\omega}^2 \overline{\psi}^\varepsilon - F(s, B., \overline{\psi}^\varepsilon, \partial_\omega \overline{\psi}^\varepsilon) \\ \geq & -\partial_t \psi^\varepsilon + \rho_0(\varepsilon) - \frac{1}{2} Id : \partial_{\omega\omega}^2 \psi^\varepsilon - F(s, B., \psi^\varepsilon, \partial_\omega \psi^\varepsilon) \\ \geq & -\partial_t \psi^\varepsilon - \frac{1}{2} Id : \partial_{\omega\omega}^2 \psi^\varepsilon - F\left(s, \sum_{i \geq -1} \hat{B}_{\cdot \wedge H_i^\varepsilon}^\varepsilon \mathbf{1}_{[H_i^\varepsilon, H_{i+1}^\varepsilon)}, \psi^\varepsilon, \partial_\omega \psi^\varepsilon\right) = 0. \end{aligned}$$

That is, $\overline{\psi}^\varepsilon \in \overline{\mathcal{D}}(0, \mathbf{0})$. Then $\overline{u}(0, \mathbf{0}) \leq \overline{\psi}^\varepsilon(0, \mathbf{0})$. Similarly, one can prove $\underline{u}(0, \mathbf{0}) \geq \underline{\psi}^\varepsilon(0, \mathbf{0})$.

Thus

$$\overline{u}(0, \mathbf{0}) - \underline{u}(0, \mathbf{0}) \leq \overline{\psi}^\varepsilon(0, \mathbf{0}) - \underline{\psi}^\varepsilon(0, \mathbf{0}) = 2\rho_0(\varepsilon)[1 + T - t].$$

Send $\varepsilon \rightarrow 0$, we obtain $\overline{u}(0, \mathbf{0}) \leq \underline{u}(0, \mathbf{0})$. This, together with (7.5), implies that $\overline{u}(0, \mathbf{0}) = \underline{u}(0, \mathbf{0})$. ■

Remark 7.5 The above proof of Proposition 7.3 takes advantage of the following three facts in the semi-linear case, which do not hold anymore in the fully nonlinear case in [10].

(i) The proof of partial comparison principle (in [8]) applies implicitly the dominated convergence theorem. In the fully nonlinear case, in order to avoid the application dominated convergence theorem, we shall modify the space $\overline{C}^{1,2}(\Lambda)$ slightly.

(ii) The functions θ_n^ε can be defined via BSDEs. In the fully nonlinear case, in particular when there is no representation formula, we shall prove the existence of such θ_n^ε satisfying (7.7) in an abstract way in [10].

(iii) The functions θ_n^ε are already in $C^{1,2}$. In the fully nonlinear case, this is typically not true, and then we shall approximate θ_n^ε by smooth functions. ■

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